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Topology

Definition

subspace

isometry

Topology

Definition

open set

open balls are open

Topology

Theorem

Proposition

unions and intersections of open sets

closed set

Topology

Definition

closed ball

closed balls are closed sets
A metric space \((X, d)\) is a set \(X\) and a function \(d : X \times X \to \mathbb{R}\) satisfying \(\forall x, y, z \in X\)

1. \(d(x, y) \geq 0\)
2. \(d(x, y) = 0 \iff x = y\)
3. \(d(x, y) = d(y, x)\)
4. \(d(x, z) \leq d(x, y) + d(y, z)\)

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Suppose \((X_1, d_1)\) and \((X_2, d_2)\) are metric spaces. A function \(f : X_1 \to X_2\) is called an isometry if \(f\) is one–to–one, onto and \(d_2(f(x), f(y)) = d_1(x, y)\ \ \forall x, y \in X_1\)

If \((X, d)\) is a metric space, and \(A \subset X\) then \((A, d|_{A \times A})\) is a metric space and is called a subspace of \((X, d)\).

Supposing \((X, d)\) is a metric space, then a subset \(U \subset X\) is open iff \(\forall x \in U, \exists r > 0\) such that \(B(x, r) \subset U\)

Let \((X, d)\) be a metric space and let \(\{U_\alpha\}_{\alpha \in A}\) be any collection of open sets in \((X, d)\), then

1. \(X, \emptyset\) are open.
2. \(\bigcup_{\alpha \in A} U_\alpha\) is open.
3. Let \(\{U_1, \ldots, U_n\}\) be a finite collection of open sets, then \(\bigcap_{i=1}^n U_i\) is open.

A closed ball centered at \(x\) of radius \(r\) is denoted \(\overline{B}(x, r)\), and defined to be:

\[
\overline{B}(x, r) = \{y \in X \mid d(x, y) \leq r\}
\]
unions and intersections of closed sets

interior

closure

exterior & frontier

distance from a point to a set

limit of a sequence

Cauchy Sequence

convergent sequence

convergence implies Cauchy

complete metric space
Let \((X,d)\) be a metric space with \(A \subset X\). The **interior** of \(A\) denoted \(A^\circ\) is defined to be:
\[
A^\circ = \{ x \in A \mid \exists r > 0 \text{ such that } B(x, r) \subset A \}
\]

Let \((X,d)\) be a metric space and let \(\{F_\alpha\}_{\alpha \in A}\) be any collection of closed sets in \((X,d)\), then
1. \(X, \emptyset\) are closed.
2. \(\bigcap_{\alpha \in A} F_\alpha\) is closed.
3. Let \(\{F_1, \ldots, F_n\}\) be a finite collection of closed sets, then \(\bigcup_{i=1}^n F_i\) is closed.

Let \((X,d)\) be a metric space with \(A \subset X\). The **exterior** of \(A\) is defined to be \((X - A)^\circ\).

The **frontier** of \(A\) is defined to be \(\overline{A} - A^\circ\).

Suppose \((X,d)\) is a metric space. A sequence \(\{x_n\} \subset X\) has **limit** \(x\), denoted \(\lim_{n \to \infty} \{x_n\} = x\) iff
\[
\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } n \geq N \Rightarrow x_n \in B(x, \varepsilon)
\]

A sequence \(\{x_n\}\) **converges** iff \(\lim \{x_n\}\) exists.

A metric space \((X,d)\) is **complete** iff every Cauchy sequence in \(X\) is convergent.

Suppose \((X,d)\) is a metric space with \(A \subset X\) and \(x \in X\). We define the distance from \(x\) to \(A\) by
\[
d(x, A) = \inf \{d(x, y) \mid y \in A\}
\]

Suppose \((X,d)\) is a metric space. A sequence \(\{x_n\} \subset X\) is called a **Cauchy sequence** iff
\[
\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } m, n \geq N \Rightarrow d(x_m, x_n) < \varepsilon
\]

A sequence \(\{x_n\}\) **converges** iff \(\lim \{x_n\}\) exists.

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A metric space \((X,d)\) is **complete** iff every Cauchy sequence in \(X\) is convergent.

If a sequence \(\{x_n\}\) is convergent then it is Cauchy.
Theorem

limits are unique
distinct points have a radius of separation

Topology

Definition

continuous function
continuous function (alternate definition)

Topology

Definition

Lipschitz function
Lipschitz functions are uniformly continuous

Topology

Definition

bi-Lipschitz

f continuous iff the preimage of every open set is open

Topology

Theorem

continuous functions and sequences
homeomorphism
Suppose $(X, d)$ is a metric space, and $x, y \in X$ with $x \neq y$, then $\exists r > 0$ such that $B(x, r) \cap B(y, r) = \emptyset$.

If the limit of $\{x_n\}$ exists, then that limit is unique.

Suppose $(X_1, d_1), (X_2, d_2)$ are metric spaces. A function $f : X_1 \rightarrow X_2$ is **continuous** on $X_1$ iff

$$\forall x \in X_1, \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } f(B(x, \delta)) \subset B(f(x), \varepsilon)$$

Suppose $(X_1, d_1), (X_2, d_2)$ are metric spaces. A function $f : X_1 \rightarrow X_2$ is **continuous** at $x \in X_1$ iff

$$\forall \varepsilon > 0, \exists \delta(x, \varepsilon) > 0 \text{ such that } d_1(x, y) < \delta \Rightarrow d_2(f(x), f(y)) < \varepsilon$$

Suppose $(X_1, d_1), (X_2, d_2)$ are metric spaces. A function $f : X_1 \rightarrow X_2$ is **Lipschitz** iff

$$\forall x, y \in X_1 \exists c > 0 \text{ such that } d_2(f(x), f(y)) \leq c d_1(x, y)$$

A Lipschitz function can be thought of as a “bounded distortion.”

A function $f : X_1 \rightarrow X_2$ is continuous iff

$$\forall U \text{ open } \subset X_2 \Rightarrow f^{-1}(U) \text{ open } \subset X_1$$

Or equivalently:

$$\forall U \text{ closed } \subset X_2 \Rightarrow f^{-1}(U) \text{ closed } \subset X_1$$

Suppose $(X_1, d_1), (X_2, d_2)$ are metric spaces. A function $f : X_1 \rightarrow X_2$ is called **bi-Lipschitz** iff

$$\forall x, y \in X_1 \exists c_1, c_2 > 0 \text{ such that } c_1 d_1(x, y) \leq d_2(f(x), f(y)) \leq c_2 d_1(x, y)$$

A function $f : (X_1, d_1) \rightarrow (X_2, d_2)$ is called a **homeomorphism** iff

1. $f$ is continuous
2. $f$ is 1-1 and onto
3. $f^{-1}$ is continuous

A function $f : (X_1, d_1) \rightarrow (X_2, d_2)$ is continuous iff

$$\forall \text{ convergent sequences } \{x_n\} \subset X_1, \lim_{n \rightarrow \infty} f(x_n) = f(\lim_{n \rightarrow \infty} \{x_n\})$$
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**Topology**

**Theorem**

composition of continuous functions preserves continuity

**Topology**

**Definition**

topology

**Topology**

**Definition**

homeomorphic spaces

**Topology**

**Definition**

topological space

**Topology**
Two metrics, $d_1$, $d_2$ are equivalent iff $id : (X, d_1) \to (X, d_2)$ is a homeomorphism.

Two metrics $d_1$, $d_2$ are called equivalent iff they have the same open sets.

Two metric spaces are homeomorphic iff there exists a homeomorphism between them.

Suppose $f : X_1 \to X_2$ and $g : X_2 \to X_3$. If $f$ and $g$ are continuous then $g \circ f$ is continuous.

Suppose $X$ is a set. A collection $\tau$ of subsets of $X$ is called a topology on $X$ iff

1. $X \in \tau$ and $\emptyset \in \tau$
2. $U_\alpha \in \tau$ for $\alpha \in A \Rightarrow \bigcup_{\alpha \in A} U_\alpha \in \tau$
3. $U_1, U_2, \ldots, U_n \in \tau \Rightarrow \bigcap_{i=1}^{\infty} U_i \in \tau$

A topological space $(X, \tau)$ is a set $X$ and a topology $\tau$ on $X$. 