Theorem 1. Let $f$ be a continuous function. If
\[ \int_0^1 f(x) \, dx \neq 0, \]
then there exists a point $x$ in the interval $[0,1]$ such that $f(x) \neq 0$.

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Theorem 2. Let $x$ be a real number. If $x > 0$, then $\frac{1}{x} > 0$.

Theorem 3. Let $x$ be a real number. If $x > 0$, then $\frac{1}{x} > 0$.

Theorem 4. Let $A$ be a set. Then $\emptyset \subseteq A$.

Theorem 5. Let $A$ and $B$ be subsets of a universal set $U$. Then $A \cap (U \setminus B) = A \setminus B$.

Theorem 7. If $A$ and $B$ are subsets of a set $U$ and $A^c$ and $B^c$ are their complements in $U$, then
\begin{enumerate}
\item $(A \cup B)^c = A^c \cap B^c$.
\item $(A \cap B)^c = A^c \cup B^c$.
\end{enumerate}

Theorem 8. $(a,b) = (c,d)$ iff $a = c$ and $b = d$.

Theorem 9. Let $R$ be an equivalence relation on a set $S$. Then $\{E_x : x \in S\}$ is a partition of $S$. The relation “belongs to the same piece as” is the same as $R$. Conversely, if $T$ is a partition of $S$, let $R$ be defined by $xRy$ iff $x$ and $y$ are in the same piece of the partition. Then $R$ is an equivalence relation and the corresponding partition into equivalence classes is the same as $T$.
Theorem 10 (part 1)

Theorem 10 (part 2)

Theorem 11

Theorem 12

Theorem 13

Theorem 14

Theorem 15

Theorem 16

Theorem 17

Theorem 18
Theorem 10. Suppose that \( f : A \to B \). Let \( C, C_1 \) and \( C_2 \) be subsets of \( A \) and let \( D, D_1 \) and \( D_2 \) be subsets of \( B \). Then the following hold:

1. \( f^{-1}(D) \subseteq f^{-1}(D_1) \cup f^{-1}(D_2) \).
2. \( f^{-1}(D_1) \cap f^{-1}(D_2) \subseteq f^{-1}(D) \).
3. \( f^{-1}(D_1) \cup f^{-1}(D_2) \subseteq f^{-1}(D_1 \cap D_2) \).
4. \( f^{-1}(D_1 \setminus D_2) = f^{-1}(D_1) \setminus f^{-1}(D_2) \) if \( D_2 \subseteq D_1 \).
5. \( f^{-1}(D_1 \setminus D_2) = f^{-1}(D_1) \setminus f^{-1}(D_2) \) if \( D_2 \supseteq D_1 \).

Theorem 12. Let \( f : A \to B \) and \( g : B \to C \). Then

1. If \( f \) and \( g \) are surjective, then \( g \circ f \) is surjective.
2. If \( f \) and \( g \) are injective, then \( g \circ f \) is injective.
3. If \( f \) and \( g \) are bijective, then \( g \circ f \) is bijective.

Theorem 14. Let \( f : A \to B \) and \( g : B \to C \) be bijective. The composition \( g \circ f : A \to C \) is bijective and \((g \circ f)^{-1} = f^{-1} \circ g^{-1}\).

Theorem 16. Let \( S \) be a nonempty set. The following three conditions are equivalent:

1. \( S \) is countable.
2. There exists an injection \( f : S \to \mathbb{N} \).
3. There exists a surjection \( f : \mathbb{N} \to S \).

Theorem 18. Let \( S, T \) and \( U \) be sets.

1. If \( S \subseteq T \), then \( |S| \leq |T| \).
2. \( |S| \leq |S| \).
3. If \( |S| \leq |T| \) and \( |T| \leq |U| \), then \( |S| \leq |U| \).
4. If \( m, n \in \mathbb{N} \) and \( m \leq n \), then \( |\{1, 2, \ldots, m\}| \leq |\{1, 2, \ldots, n\}| \).
5. If \( S \) is finite, then \( S < \aleph_0 \).

Theorem 17. The set \( \mathbb{R} \) of real numbers is uncountable.
Theorem 19

Theorem 19

Principle of Mathematical Induction

Real Analysis I

Real Analysis I

Theorem 20

Theorem 20

Theorem 21

Theorem 21

The Binomial Formula

Real Analysis I

Real Analysis I

Theorem 22

Theorem 22

Real Analysis I

Real Analysis I

Theorem 23

Theorem 23

The Binomial Formula

Real Analysis I

Real Analysis I

Theorem 24

Theorem 24

Real Analysis I

Real Analysis I

Theorem 25

Theorem 25

Real Analysis I

Real Analysis I

Theorem 26

Theorem 26

Real Analysis I

Real Analysis I

Theorem 27

Theorem 27

Real Analysis I

Real Analysis I

Theorem 28

Theorem 28

Real Analysis I

Real Analysis I
Theorem 20. (Principle of Mathematical Induction) Let $P(n)$ be a statement that is either true or false for each $n \in \mathbb{N}$. Then $P(n)$ is true for all $n \in \mathbb{N}$ provided that

1. $P(1)$ is true, and
2. for each $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k + 1)$ is true.

Theorem 22. $7^n - 4^n$ is a multiple of 3 for all $n \in \mathbb{N}$.

Theorem 24. Let $m \in \mathbb{N}$ and let $P(n)$ be a statement that is either true or false for each $n \geq m$. Then $P(n)$ is true for all $n \geq m$ provided that

1. $P(m)$ is true, and
2. for each $k \geq m$, if $P(k)$ is true, then $P(k + 1)$ is true.

Theorem 26. Let $x, y \in \mathbb{R}$ such that $x \leq y + \epsilon$ for every $\epsilon > 0$. Then $x \leq y$.

Theorem 28. Let $m, n, p \in \mathbb{Z}$. If $p$ is a prime number and $p$ divides the product $mn$, then $p$ divides $m$ or $p$ divides $n$.

Theorem 19. For any set $S$, we have $|S| < |\mathcal{P}(S)|$.

Theorem 21. $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$ for every natural number $n$.

Theorem 23. (The Binomial Formula) If $x$ and $y$ are real numbers and $n \in \mathbb{N}$, then

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$
Theorem 29

Real Analysis I

Theorem 31

Real Analysis I

Theorem 33

Real Analysis I

Theorem 35

Archimedean Property of \( \mathbb{R} \)

Real Analysis I

Theorem 37

Real Analysis I
Theorem 30. Every non-empty subset of $\mathbb{R}$ that is bounded below has a greatest lower bound.

Theorem 29. Let $p$ be a prime number. Then $\sqrt{p}$ is not a rational number.

Theorem 32. Let $A$ and $B$ be non-empty subsets of $\mathbb{R}$. Then

1. $\inf A \leq \sup A$.
2. $\sup(-A) = -\inf A$ and $\inf(-A) = -\sup A$.
3. $\sup(A+B) = \sup(A) + \sup(B)$ and $\inf(A+B) = \inf(A) + \inf(B)$.
4. $\sup(A-B) = \sup(A) - \inf(B)$.
5. If $A \subseteq B$, then $\sup A \leq \sup B$ and $\inf B \leq \inf A$.

Theorem 34. Let $f$ and $g$ be functions defined on a set containing $A$ as a subset, and let $c \in \mathbb{R}$ be a positive constant. Then

1. $\sup_A cf = c \sup_A f$ and $\inf_A cf = c \inf_A f$.
2. $\sup_A (-f) = -\inf_A f$.
3. $\sup_A (f+g) \leq \sup_A f + \sup_A g$ and $\inf_A f + \inf_A g \leq \inf_A (f+g)$.
4. $\sup \{f(x) - f(y) : x, y \in A\} \leq \sup_A f - \inf_A f$.

Theorem 36. (Archimedean Property of $\mathbb{R}$) The set $\mathbb{N}$ of natural numbers is unbounded above in $\mathbb{R}$.

Theorem 35. The real number system $\mathbb{R}$ is a complete ordered field.

Theorem 38. Let $p$ be a prime number. Then there exists a positive real number $x$ such that $x^2 = p$.

Theorem 31. Let $A$ be a non-empty subset of $\mathbb{R}$ and $x$ an element of $\mathbb{R}$. Then

1. $\sup A \leq x$ iff $a \leq x$ for every $a \in A$.
2. $x < \sup A$ iff $x < a$ for some $a \in A$.

Theorem 33. Suppose that $D$ is a nonempty set and that $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$. If for every $x, y \in D$, $f(x) \leq g(y)$, then $f(D)$ is bounded above and $g(D)$ is bounded below. Furthermore, $\sup f(D) \leq \sup g(D)$.

Theorem 37. Each of the following is equivalent to the Archimedean property.

1. For each $z \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > z$.
2. For each $x > 0$ and for each $y \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $nx > y$.
3. For each $x > 0$, there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < x$. 

Theorem 39

Real Analysis I

Theorem 40

Real Analysis I

Theorem 41

Real Analysis I

Corollary 1

Real Analysis I

Theorem 43

Real Analysis I

Lemma 1

Real Analysis I

Theorem 44

Heine–Borel Theorem

Real Analysis I

Theorem 45

Bolzano–Weierstrass Theorem

Real Analysis I

Theorem 46

Real Analysis I
Theorem 40. If \( x \) and \( y \) are real numbers with \( x < y \), then there exists an irrational number \( w \) such that \( x < w < y \).

Theorem 39. (Density of \( \mathbb{Q} \) in \( \mathbb{R} \)) If \( x \) and \( y \) are real numbers with \( x < y \), then there exists a rational number \( r \) such that \( x < r < y \).

Theorem 42.

1. The union of any collection of open sets is an open set.
2. The intersection of any finite collection of open sets is an open set.

Theorem 41.

1. A set \( S \) is open iff \( S = \text{int} \ S \). Equivalently, \( S \) is open iff every point in \( S \) is an interior point of \( S \).
2. A set \( S \) is closed iff its complement \( \mathbb{R} \setminus S \) is open.

Theorem 43. Let \( S \) be a subset of \( \mathbb{R} \). Then

1. \( S \) is closed iff \( S \) contains all of its accumulation points.
2. \( \text{cl} \ S \) is a closed set.
3. \( S \) is closed iff \( S = \text{cl} \ S \).

Theorem 44. (Heine–Borel) A subset \( S \) of \( \mathbb{R} \) is compact iff \( S \) is closed and bounded.

Corollary 1.

1. The intersection of any collection of closed sets is closed.
2. The union of any finite collection of closed sets is closed.

Theorem 46. (Heine–Borel) A subset \( S \) of \( \mathbb{R} \) is compact iff \( S \) is closed and bounded.

Lemma 1. If \( S \) is a nonempty closed bounded subset of \( \mathbb{R} \), then \( S \) has a maximum and a minimum.

Theorem 45. (Bolzano–Weierstrass) If a bounded subset \( S \) of \( \mathbb{R} \) contains infinitely many points, then there exists at least one point in \( \mathbb{R} \) that is an accumulation point of \( S \).
Corollary 2
Nested Intervals Theorem

Theorem 47

Theorem 48

Theorem 49

Theorem 50

Theorem 51

Theorem 52
The Squeeze Principle

Theorem 53

Theorem 54

Corollary 3
**Theorem 47.** Let \((s_n)\) and \((a_n)\) be sequences of real numbers and let \(s \in \mathbb{R}\). If for some \(k > 0\) and some \(m \in \mathbb{N}\), we have
\[
|s_n - s| \leq k|a_n|,
\]
for all \(n > m\), and if \(\lim a_n = 0\), then it follows that \(\lim s_n = s\).

**Corollary 2.** *(Nested Intervals Theorem)* Let \(\mathcal{F} = \{A_n : n \in \mathbb{N}\}\) be a family of closed bounded intervals in \(\mathbb{R}\) such that \(A_{n+1} \subseteq A_n\) for all \(n \in \mathbb{N}\). Then \(\bigcap_{n=1}^{\infty} A_n \neq \emptyset\).

**Theorem 49.** If a sequence converges, its limit is unique.

**Theorem 48.** Every convergent sequence is bounded.

**Theorem 51.** Let \((s_n)\) be a sequence of real numbers such that \(\lim s_n = 0\), and let \((t_n)\) be a bounded sequence. Then \(\lim s_n t_n = 0\).

**Theorem 50.** A sequence \((s_n)\) converges to \(s\) iff for each \(\epsilon > 0\), there are only finitely many \(n\) for which \(|s_n - s| \geq \epsilon\).

**Theorem 53.** Suppose that \((s_n)\) and \((t_n)\) are convergent sequences with \(\lim s_n = s\) and \(\lim t_n = t\). Then

1. \(\lim(s_n + t_n) = s + t\).
2. \(\lim(ks_n) = ks\) and \(\lim(k + s_n) = k + s\) for any \(k \in \mathbb{R}\).
3. \(\lim(s_n t_n) = st\).
4. \(\lim \left(\frac{s_n}{t_n}\right) = \frac{s}{t}\), provided that \(t_n \neq 0\) for all \(n\) and \(t \neq 0\).

**Corollary 3.** If \((t_n)\) converges to \(t\) and \(t_n \geq 0\) for all \(n \in \mathbb{N}\), then \(t \geq 0\).

**Theorem 52.** *(The Squeeze Principle)* If \((a_n)\), \((b_n)\), and \((c_n)\) are sequences for which there is a number \(K\) such that \(b_n \leq a_n \leq c_n\) for all \(n > K\), and if \(b_n \to a\) and \(c_n \to a\), then \(a_n \to a\).

**Theorem 54.** Suppose that \((s_n)\) and \((t_n)\) are convergent sequences with \(\lim s_n = s\) and \(\lim t_n = t\). If \(s_n \leq t_n\) for all \(n \in \mathbb{N}\), then \(s \leq t\).
Theorem 55
Ratio Test

Real Analysis I

Theorem 56

Real Analysis I

Theorem 57
Monotone Convergence Theorem

Real Analysis I

Theorem 58

Real Analysis I

Theorem 59

Real Analysis I

Lemma 2

Real Analysis I

Theorem 60
Cauchy Convergence Criterion

Real Analysis I

Theorem 61

Real Analysis I

Bolzano–Weierstrass Theorem For Sequences

Real Analysis I
Theorem 56. Suppose that \((s_n)\) and \((t_n)\) are sequences such that \(s_n \leq t_n\) for all \(n \in \mathbb{N}\).

1. If \(\lim s_n = +\infty\), then \(\lim t_n = +\infty\).
2. If \(\lim t_n = -\infty\), then \(\lim s_n = -\infty\).

Theorem 55. (Ratio Test) Suppose that \((s_n)\) is a sequence of positive terms and that the limit \(L = \lim \left(\frac{s_{n+1}}{s_n}\right)\) exists. If \(L < 1\), then \(\lim s_n = 0\).

Theorem 58. (Monotone Convergence Theorem) A monotone sequence is convergent iff it is bounded.

Theorem 57. Let \((s_n)\) be a sequence of positive numbers. Then \(\lim s_n = +\infty\) iff \(\lim \left(\frac{1}{s_n}\right) = 0\).

Theorem 59.

1. If \((s_n)\) is an unbounded increasing sequence, then \(\lim s_n = +\infty\).
2. If \((s_n)\) is an unbounded decreasing sequence, then \(\lim s_n = -\infty\).

Theorem 60. (Cauchy Convergence Criterion) A sequence of real numbers is convergent iff it is a Cauchy sequence.

Lemma 3. Every Cauchy sequence is bounded.

Theorem 62. (Bolzano–Weierstrass Theorem For Sequences) Every bounded sequence has a convergent subsequence.

Theorem 61. If a sequence \((s_n)\) converges to a real number \(s\), then every subsequence of \((s_n)\) also converges to \(s\).
Theorem 63

Real Analysis I

Theorem 64

Real Analysis I

Theorem 65

Real Analysis I

Theorem 66

Real Analysis I

Corollary 4

Real Analysis I

Theorem 67

Real Analysis I

Theorem 68

Real Analysis I

Theorem 69

Real Analysis I

Theorem 70

Real Analysis I

Theorem 71

Real Analysis I
Theorem 64. Let \((s_n)\) be a sequence and suppose that \(m = \lim s_n\) is a real number. Then the following properties hold:

1. For every \(\epsilon > 0\) there exists \(N\) such that \(n > N\) implies that \(s_n < m + \epsilon\).
2. For every \(\epsilon > 0\) and for every \(i \in \mathbb{N}\), there exists an integer \(k > i\) such that \(s_k > m - \epsilon\).

Theorem 66. Let \(f : D \rightarrow \mathbb{R}\) and let \(c\) be an accumulation point of \(D\). Then \(\lim_{x \to c} f(x) = L\) iff for every sequence \((s_n)\) in \(D\) that converges to \(c\) with \(s_n \neq c\) for all \(n\), the sequence \((f(s_n))\) converges to \(L\).

Theorem 67. Let \(f : D \rightarrow \mathbb{R}\) and let \(c\) be an accumulation point of \(D\). Then the following are equivalent:

(a) \(f\) does not have a limit at \(c\).
(b) There exists a sequence \((s_n)\) in \(D\) with each \(s_n\) \(\neq c\) such that \((s_n)\) converges to \(c\), but \((f(s_n))\) is not convergent in \(\mathbb{R}\).

Theorem 69. Let \(f : D \rightarrow \mathbb{R}\) and let \(c \in D\). Then the following three conditions are equivalent:

(a) \(f\) is continuous at \(c\).
(b) If \((x_n)\) is any sequence in \(D\) such that \((x_n)\) converges to \(c\), then \(\lim f(x_n) = f(c)\).
(c) For every neighborhood \(V\) of \(f(c)\) there exists a neighborhood \(U\) of \(c\) such that \(f(U \cap D) \subseteq V\).

Furthermore, if \(c\) is an accumulation point of \(D\), then the above are all equivalent to

(d) \(f\) has a limit at \(c\) and \(\lim_{x \to c} f(x) = f(c)\).

Theorem 70. Let \(f : D \rightarrow \mathbb{R}\) and let \(c \in D\). Then \(f\) is discontinuous at \(c\) iff there exists a sequence \((x_n)\) in \(D\) such that \((x_n)\) converges to \(c\) but the sequence \((f(x_n))\) does not converge to \(f(c)\).

Theorem 71. Let \(f\) and \(g\) be functions from \(D\) to \(\mathbb{R}\), and let \(c \in D\). Suppose that \(f\) and \(g\) are continuous at \(c\). Then

(a) \(f + g\) and \(fg\) are continuous at \(c\),
(b) \(f/g\) is continuous at \(c\) if \(g(c) \neq 0\).

Theorem 63. Every unbounded sequence contains a monotone subsequence that has either \(+\infty\) or \(-\infty\) as a limit.
Theorem 72

Real Analysis I

Corollary 5

Real Analysis I

Theorem 74

Intermediate Value Theorem

Real Analysis I

Theorem 76

Real Analysis I

Theorem 78

Real Analysis I
**Theorem 73.** Let $D$ be a compact subset of $\mathbb{R}$ and suppose that $f : D \to \mathbb{R}$ is continuous. Then $f(D)$ is compact.

**Theorem 72.** Let $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be functions such that $f(D) \subseteq E$. If $f$ is continuous at a point $c \in D$ and $g$ is continuous at $f(c)$, then the composition $g \circ f : D \to \mathbb{R}$ is continuous at $c$.

**Lemma 4.** Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose that $f(a) < 0 < f(b)$. Then there exists a point $c$ in $(a, b)$ such that $f(c) = 0$.

**Corollary 5.** Let $D$ be a compact subset of $\mathbb{R}$ and suppose that $f : D \to \mathbb{R}$ is continuous. Then $f$ assumes minimum and maximum values on $D$. That is, there exist points $x_1$ and $x_2$ in $D$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in D$.

**Theorem 75.** Let $I$ be a compact interval and suppose that $f : I \to \mathbb{R}$ is a continuous function. Then the set $f(I)$ is a compact interval.

**Theorem 74.** (Intermediate Value Theorem) Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Then $f$ has the intermediate value property on $[a, b]$. That is, if $k$ is any value between $f(a)$ and $f(b)$ [i.e. $f(a) < k < f(b)$ or $f(b) < k < f(a)$], then there exists $c \in [a, b]$ such that $f(c) = k$.

**Theorem 77.** Let $f : D \to \mathbb{R}$ be uniformly continuous on $D$ and suppose that $(x_n)$ is a Cauchy sequence in $D$. Then $(f(x_n))$ is a Cauchy sequence.

**Theorem 76.** Suppose that $f : D \to \mathbb{R}$ is continuous on a compact set $D$. Then $f$ is uniformly continuous on $D$.

**Theorem 79.** Let $I$ be an interval containing the point $c$ and suppose that $f : I \to \mathbb{R}$. Then $f$ is differentiable at $c$ iff, for every sequence $(x_n)$ in $I \setminus \{c\}$ that converges to $c$, the sequence

$$
\left( \frac{f(x_n) - f(c)}{x_n - c} \right)
$$

converges. Furthermore, if $f$ is differentiable at $c$, then the sequence of quotients above will converge to $f'(c)$.

**Theorem 78.** A function $f : (a, b) \to \mathbb{R}$ is uniformly continuous on $(a, b)$ iff it can be extended to a function $\tilde{f}$ that is continuous on $[a, b]$.
Theorem 80

Theorem 81 (part 1)

Real Analysis I

Theorem 81 (part 2)

Theorem 82

Chain Rule

Real Analysis I

Theorem 83

Theorem 84

Rolle’s Theorem

Real Analysis I

Theorem 85

Mean Value Theorem

Theorem 86

Real Analysis I

Corollary 6

Theorem 87

Real Analysis I
**Theorem 81.** Suppose that $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are differentiable at $c \in I$. Then

(a) If $k \in \mathbb{R}$, then the function $kf$ is differentiable at $c$ and $(kf)'(c) = k \cdot f'(c)$.

(b) The function $f + g$ is differentiable at $c$ and $(f + g)'(c) = f'(c) + g'(c)$.

**Theorem 82.** (Chain Rule) Let $I$ and $J$ be intervals in $\mathbb{R}$, let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$, where $f(I) \subseteq J$, and let $c \in I$. If $f$ is differentiable at $c$ and $g$ is differentiable at $f(c)$, then the composite function $g \circ f$ is differentiable at $c$ and $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.

**Theorem 84.** (Rolle’s Theorem) Let $f$ be a continuous function on $[a, b]$ that is differentiable on $(a, b)$ and such that $f(a) = f(b) = 0$. Then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.

**Theorem 86.** Let $f$ be continuous on $[a, b]$ and differentiable on $(a, b)$. If $f'(x) = 0$ for all $x \in (a, b)$, then $f$ is constant on $[a, b]$.

**Theorem 88.** Suppose that $f : I \to \mathbb{R}$ is differentiable at a point $c \in I$, then $f$ is continuous at $c$.

**Theorem 81.** Suppose that $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are differentiable at $c \in I$. Then

(c) (Product Rule) The function $fg$ is differentiable at $c$ and $(fg)'(c) = f(c)g'(c) + f'(c)g(c)$.

(d) (Quotient Rule) If $g(c) \neq 0$, then the function $f/g$ is differentiable at $c$ and

$$
\left( \frac{f}{g} \right)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}.
$$

**Theorem 83.** If $f$ is differentiable on an open interval $(a, b)$ and if $f$ assumes its maximum or minimum at a point $c \in (a, b)$, then $f'(c) = 0$.

**Theorem 85.** (Mean Value Theorem) Let $f$ be a continuous function on $[a, b]$ that is differentiable on $(a, b)$. Then there exists at least one point $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

**Corollary 6.** Let $f$ and $g$ be continuous on $[a, b]$ and differentiable on $(a, b)$. Suppose that $f'(x) = g'(x)$ for all $x \in (a, b)$. Then there exists a constant $C$ such that $f = g + C$ on $[a, b]$. 
Theorem 88
Intermediate Value Theorem for Derivatives

Theorem 89
Inverse Function Theorem

Real Analysis I

Theorem 90
Cauchy Mean Value Theorem

Theorem 91
L’Hospital’s Rule

Real Analysis I

Theorem 92
L’Hospital’s Rule

Theorem 93
Taylor’s Theorem

Real Analysis I

Theorem 94

Real Analysis I

Theorem 95

Real Analysis I

Theorem 96

Real Analysis I

Theorem 97

Real Analysis I
Theorem 89. (Inverse Function Theorem) Suppose that \( f \) is differentiable on an interval \( I \) and \( f'(x) \neq 0 \) for all \( x \in I \). Then \( f \) is injective, \( f^{-1} \) is differentiable on \( f(I) \), and \( (f^{-1})'(y) = \frac{1}{f'(x)} \), where \( y = f(x) \).

Theorem 90. (Cauchy Mean Value Theorem) Let \( f \) and \( g \) be functions that are continuous on \([a, b]\) and differentiable on \((a, b)\). Then there exists at least one point \( c \in (a, b) \) such that

\[
[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)
\]

Theorem 91. (L'Hospital's Rule) Let \( f \) and \( g \) be continuous on \([a, b]\) and differentiable on \((a, b)\). Suppose that \( c \in [a, b] \) and \( f(c) = g(c) = 0 \). Suppose also that \( g'(x) \neq 0 \) for \( x \in U \), where \( U \) is the intersection of \((a, b)\) and some deleted neighborhood of \( c \). If

\[
\lim_{x \to c} \frac{f'(x)}{g'(x)} = L, \quad \text{with} \quad L \in \mathbb{R},
\]

then

\[
\lim_{x \to c} \frac{f(x)}{g(x)} = L.
\]

Theorem 92. (L'Hospital's Rule) Let \( f \) and \( g \) be differentiable on \((b, \infty)\). Suppose that \( \lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = \infty \), and that \( g'(x) \neq 0 \) for \( x \in (b, \infty) \).

If \( \lim_{x \to \infty} \frac{f'(x)}{g'(x)} = L \), where \( L \in \mathbb{R} \), then

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = L.
\]

Theorem 93. (Taylor's Theorem) Let \( f \) and its first \( n \) derivatives be continuous on \([a, b]\) and differentiable on \((a, b)\), and let \( x_0 \in [a, b] \). Then for each \( x \in [a, b] \) with \( x \neq x_0 \) there exists a point \( c \) between \( x \) and \( x_0 \) such that

\[
f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots
\]

\[
+ \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.
\]

Theorem 94. Let \( f \) be a bounded function on \([a, b]\). If \( P \) and \( Q \) are partitions of \([a, b]\) and \( Q \) is a refinement of \( P \), then \( L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P) \).

Theorem 95. Let \( f \) be a bounded function on \([a, b]\). Then \( L(f) \leq U(f) \).

Theorem 96. Let \( f \) be a bounded function on \([a, b]\). Then \( f \) is integrable iff for each \( \epsilon > 0 \) there exists a partition \( P \) of \([a, b]\) such that \( U(f, P) - L(f, P) < \epsilon \).

Theorem 97. Let \( f \) be a monotonic function on \([a, b]\). Then \( f \) is integrable.
Theorem 98

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Theorem 99

Real Analysis I

Theorem 100

Real Analysis I

Corollary 7

Real Analysis I

Theorem 101

The Fundamental Theorem of Calculus I

Real Analysis I

Theorem 102

The Fundamental Theorem of Calculus II

Real Analysis I

Theorem 103

Cauchy Criterion for Series

Real Analysis I
Theorem 99. Let \( f \) and \( g \) be integrable functions on \([a, b] \) and let \( k \in \mathbb{R} \). Then

(a) \( kf \) is integrable and \( \int_a^b kf = k \int_a^b f \), and

(b) \( f + g \) is integrable and \( \int_a^b (f + g) = \int_a^b f + \int_a^b g \).

Theorem 98. Let \( f \) be a continuous function on \([a, b] \). Then \( f \) is integrable on \([a, b] \).

Theorem 101. Suppose that \( f \) is integrable on \([a, b] \) and \( g \) is continuous on \([c, d] \), where \( f([a, b]) \subseteq [c, d] \). Then \( g \circ f \) is integrable on \([a, b] \).

Theorem 100. Suppose that \( f \) is integrable on both \([a, c] \) and \([c, b] \). Then \( f \) is integrable on \([a, b] \). Furthermore, \( \int_a^b f = \int_a^c f + \int_c^b f \).

Theorem 102. (The Fundamental Theorem of Calculus I) Let \( f \) be integrable on \([a, b] \). For each \( x \in [a, b] \) let \( F(x) = \int_a^x f(t) \, dt \). Then \( F \) is uniformly continuous on \([a, b] \). Furthermore, if \( f \) is continuous at \( c \in [a, b] \), then \( F \) is differentiable at \( c \) and \( F'(c) = f(c) \).

Corollary 7. Let \( f \) be integrable on \([a, b] \). The \( |f| \) is integrable on \([a, b] \) and \( \left| \int_a^b f \right| \leq \int_a^b |f| \).

Theorem 103. (The Fundamental Theorem of Calculus II) If \( f \) is differentiable on \([a, b] \) and \( f' \) is integrable on \([a, b] \), then \( \int_a^b f' = f(b) - f(a) \).

Theorem 104. Suppose that \( \sum a_n = s \) and \( \sum b_n = t \). Then \( \sum (a_n + b_n) = s + t \) and \( \sum (ka_n) = ks \), for every \( k \in \mathbb{R} \).

Theorem 105. If \( \sum a_n \) is a convergent series, then \( \lim a_n = 0 \).

Theorem 106. (Cauchy Criterion for Series) The infinite series \( \sum a_n \) converges iff for each \( \epsilon > 0 \) there exists a number \( N \) such that if \( n \geq m > N \), then \( |a_m + a_{m+1} + \cdots + a_n| < \epsilon \).
Theorem 107  
Comparison Test  
Real Analysis I

Theorem 108  
Real Analysis I

Theorem 109  
Ratio Test  
Real Analysis I

Theorem 110  
Root Test  
Real Analysis I

Theorem 111  
Integral Test  
Real Analysis I

Theorem 112  
Alternating Series Test  
Real Analysis I

Theorem 113  
Ratio Criterion  
Real Analysis I

Theorem 114  
Real Analysis I

Theorem 115  
Weierstrass M-test  
Real Analysis I

Theorem 116  
Real Analysis I
Theorem 108. If a series converges absolutely, then it converges.

Theorem 110. (Root Test) Given a series $\sum a_n$, let $\alpha = \limsup |a_n|^{\frac{1}{n}}$.

1. If $\alpha < 1$, then the series converges absolutely.
2. If $\alpha > 1$, then the series diverges.
3. Otherwise, $\alpha = 1$ and the test gives no information about convergence or divergence.

Theorem 112. (Alternating Series Test) If $(a_n)$ is a decreasing sequence of positive numbers and $\lim_{n \to \infty} a_n = 0$, then the series $\sum (-1)^{n+1}a_n$ converges.

Theorem 114. (Ratio Criterion) The radius of convergence $R$ of a power series $\sum a_nx^n$ is equal to $\lim \left| \frac{a_n}{a_{n+1}} \right|$, provided that this limit exists.

Theorem 116. (Weierstrass M-test) Suppose that $(f_n)$ is a sequence of functions defined on $S$ and $(M_n)$ is a sequence of nonnegative numbers such that $|f_n(x)| \leq M_n$ for all $x \in S$ and all $n \in \mathbb{N}$. If $\sum M_n$ converges, then $\sum f_n$ converges uniformly on $S$.

Theorem 107. (Comparison Test) Let $\sum a_n$ and $\sum b_n$ be infinite series of nonnegative terms. That is, $a_n \geq 0$ and $b_n \geq 0$ for all $n$. Then

1. If $\sum a_n$ converges and $0 \leq b_n \leq a_n$ for all $n$, then $\sum b_n$ converges.
2. If $\sum a_n = +\infty$ and $0 \leq a_n \leq b_n$ for all $n$, then $\sum b_n = +\infty$.

Theorem 109. (Ratio Test) Let $\sum a_n$ be a series of nonzero terms.

1. If $\limsup \left| \frac{a_{n+1}}{a_n} \right| < 1$, then the series converges absolutely.
2. If $\liminf \left| \frac{a_{n+1}}{a_n} \right| > 1$, then the series diverges.
3. Otherwise, $\liminf \left| \frac{a_{n+1}}{a_n} \right| \leq 1 \leq \limsup \left| \frac{a_{n+1}}{a_n} \right|$ and the test gives no information about convergence or divergence.

Theorem 111. (Integral Test) Let $f$ be a continuous function defined on $[0, \infty)$, and suppose that $f$ is positive and decreasing. That is, if $x_1 < x_2$, then $f(x_1) \geq f(x_2) > 0$. Then the series $\sum f(n)$ converges iff $\lim_{n \to \infty} \left( \int_1^n f(x) \, dx \right)$ exists as a real number.

Theorem 113. Let $\sum a_n x^n$ be a power series and let $\alpha = \limsup |a_n|^{\frac{1}{n}}$. Define $R$ by

$$R = \begin{cases} \frac{1}{\alpha} & \text{if } 0 < \alpha < +\infty \\ +\infty & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha = +\infty \end{cases}.$$  

Then the series converges absolutely whenever $|x| < R$ and diverges whenever $|x| > R$. (When $R = +\infty$ we take this to mean that the series converges absolutely for all real $x$. When $R = 0$ then the series converges only at $x = 0$.)

Theorem 115. Let $(f_n)$ be a sequence of functions defined on a subset $S$ of $\mathbb{R}$. There exists a function $f$ such that $(f_n)$ converges to $f$ uniformly on $S$ iff the following condition (called the Cauchy criterion) is satisfied:

For every $\epsilon > 0$ there exists a number $N$ such that $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in S$ and all $m, n > N$. 

Theorem 110. (Root Test) Given a series $\sum a_n$, let $\alpha = \limsup |a_n|^\frac{1}{n}$.

1. If $\alpha < 1$, then the series converges absolutely.
2. If $\alpha > 1$, then the series diverges.
3. Otherwise, $\alpha = 1$ and the test gives no information about convergence or divergence.
Theorem 117

Corollary 8

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Theorem 118

Corollary 9

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Theorem 119

Corollary 10

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Theorem 120

Theorem 121

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Theorem 122

Corollary 11

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Real Analysis I
Corollary 8. Let $\sum_{n=0}^{\infty} f_n$ be a series of functions defined on a set $S$. Suppose that each $f_n$ is continuous on $S$ and that the series converges uniformly to a function $f$ on $S$. Then $f = \sum_{n=0}^{\infty} f_n$ is continuous on $S$.

Corollary 9. Let $\sum_{n=0}^{\infty} f_n$ be a series of functions defined on an interval $[a, b]$. Suppose that each $f_n$ is continuous on $[a, b]$ and that the series converges uniformly to a function $f$ on $[a, b]$. Then $\int_{a}^{b} f(x) \, dx = \sum_{n=0}^{\infty} \int_{a}^{b} f_n(x) \, dx$.

Corollary 10. Let $\sum_{n=0}^{\infty} f_n$ be a series of functions that converges to a function $f$ on an interval $[a, b]$. Suppose that for each $n$, $f'_n$ exists and is continuous on $[a, b]$ and that the series of derivatives $\sum_{n=0}^{\infty} f'_n$ is uniformly convergent on $[a, b]$. Then $f'(x) = \sum_{n=0}^{\infty} f'_n(x)$ for all $x \in [a, b]$.

Theorem 117. Let $(f_n)$ be a sequence of continuous functions defined on a set $S$ and suppose that $(f_n)$ converges uniformly on $S$ to a function $f : S \to \mathbb{R}$. Then $f$ is continuous on $S$.

Theorem 118. Let $(f_n)$ be a sequence of continuous functions defined on an interval $[a, b]$ and suppose that $(f_n)$ converges uniformly on $[a, b]$ to a function $f$. Then $\lim_{n \to \infty} \int_{a}^{b} f_n(x) \, dx = \int_{a}^{b} f(x) \, dx$.

Theorem 119. Suppose that $(f_n)$ converges to $f$ on an interval $[a, b]$. Suppose also that each $f'_n$ exists and is continuous on $[a, b]$, and that the sequence $(f'_n)$ converges uniformly on $[a, b]$. Then $\lim_{n \to \infty} f'_n(x) = f'(x)$ for each $x \in [a, b]$.

Theorem 120. There exists a continuous function defined on $\mathbb{R}$ that is nowhere differentiable.

Theorem 121. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R$, where $0 < R \leq +\infty$. If $0 < K < R$, then the power series converges uniformly on $[-K, K]$.

Theorem 122. Suppose that a power series converges to a function $f$ on $(-R, R)$, where $R > 0$. Then the series can be differentiated term by term, and the differentiated series converges on $(-R, R)$ to $f'$. That is, if $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then $f'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$, and both series have the same radius of convergence.

Corollary 11. Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $x \in (-R, R)$, where $R > 0$. Then for each $k \in \mathbb{N}$, the $k$th derivative $f^{(k)}$ of $f$ exists on $(-R, R)$ and

$$f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k} = k! a_k + (k+1)! a_{k+1} + \frac{(k+2)!}{2!} a_{k+2} x^2 + \cdots$$

Furthermore, $f^{(k)}(0) = k! a_k$. 
Corollary

Corollary 12

Theorem

Theorem 123

Real Analysis I

Corollary

Corollary 13

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Theorem 123. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with a finite positive radius of convergence $R$. If the series converges at $x = R$, then it converges uniformly on the interval $[0, R]$. Similarly, if the series converges at $x = -R$, then it converges uniformly on $[-R, 0]$.

Corollary 12. If $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ for all $x$ in some interval $(-R, R)$, where $R > 0$, then $a_n = b_n$ for all $n \in \mathbb{N} \cup \{0\}$.

Corollary 13. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ have a finite positive radius of convergence $R$. If the series converges at $x = R$, then $f$ is continuous at $x = R$. If the series converges at $x = -R$, then $f$ is continuous at $x = -R$. 