Definition

sentential connectives

negation

Definition

conjunction

disjunction

Definition

implication or conditional

antecedant & consequent

hypothesis & conclusion

Definition

equivalence

negation of a conjunction
A sentence that can unambiguously be classified as true or false.

Let $p$ stand for a statement, then $\sim p$ (read not $p$) represents the logical opposite or negation of $p$.

If $p$ and $q$ are statements, then the statement $p$ or $q$ (called the disjunction of $p$ and $q$ and denoted $p \lor q$) is true unless both $p$ and $q$ are false.

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If $p$ and $q$ are statements, then the statement $p$ and $q$ (called the conjunction of $p$ and $q$ and denoted $p \land q$) is true only when both $p$ and $q$ are true, and false otherwise.

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A statement of the form 

if $p$ then $q$

is called an implication or conditional.

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A statement of the form “$p$ if and only if $q$” is the conjunction of two implications and is called an equivalence.

\[
\sim (p \land q) \iff (\sim p) \lor (\sim q)
\]
Definition

negation of a disjunction

Definition

negation of an implication

Real Analysis I

Definition

tautology

Definition

universal quantifier

Real Analysis I

Definition

existential quantifier

Definition

contrapositive

Real Analysis I

Definition

converse

Definition

inverse

Real Analysis I

Definition

contradiction

Definition

subset

Real Analysis I
\[\sim (p \Rightarrow q) \Leftrightarrow p \land (\sim q)\]

\[\sim (p \lor q) \Leftrightarrow (\sim p) \land (\sim q)\]

\[\forall x, \ p(x)\]

In the above statement, the **universal quantifier** denoted by \(\forall\) is read “for all”, “for each”, or “for every”.

A sentence whose truth table contains only \(T\) is called a **tautology**. The following sentences are examples of tautologies (\(c \equiv\) contradiction):

\[(p \Leftrightarrow q) \Leftrightarrow (p \Rightarrow q) \land (q \Rightarrow p)\]

\[(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)\]

\[(p \Rightarrow q) \Leftrightarrow [(p \land \sim q) \Rightarrow c]\]

\[\exists x \ni p(x)\]

In the above statement, the **existential quantifier** denoted by \(\exists\) is read “there exists . . .”, “there is at least one . . .”. The symbol \(\ni\) is just shorthand for “such that”.

The implication \(p \Rightarrow q\) is logically equivalent with its **contrapositive**:

\[\sim q \Rightarrow \sim p\]

Given the implication \(p \Rightarrow q\) then its **inverse** is

\[\sim p \Rightarrow \sim q\]

An implication is *not* logically equivalent to its inverse. The inverse is the contrapositive of the converse.

Let \(A\) and \(B\) be sets. We say that \(A\) is a **subset** of \(B\) if every element of \(A\) is an element of \(B\). In symbols, this is denoted

\[A \subseteq B\]

A **contradiction** is a statement that is always false. Contradictions are symbolized by the letter \(c\) or by two arrows pointing directly at each other.

Given the implication \(p \Rightarrow q\) then its **converse** is

\[q \Rightarrow p\]

But they are *not* logically equivalent.
Definition

proper subset

set equality

Definition

union, intersection, complement, disjoint

indexed family of sets

Definition

pairwise disjoint

ordered pair

Definition

Cartesian product

relation

Definition

equivalence relation

equivalence class
Let $A$ and $B$ be sets. We say that $A$ is a **equal** to $B$ if $A$ is a subset of $B$ and $B$ is a subset of $A$.

$A = B \iff A \subseteq B$ and $B \subseteq A$

Let $A$ and $B$ be sets. $A$ is a **proper subset** of $B$ if $A$ is a subset of $B$ and there exists an element in $B$ that is not in $A$.

If for each element $j$ in a nonempty set $J$ there corresponds a set $A_j$, then

$$
\mathcal{A} = \{A_j : j \in J\}
$$

is called an **indexed family of sets** with $J$ as the index set.

The **ordered pair** $(a, b)$ is the set whose members are \{a\} and \{a, b\}.

$$(a, b) = \{\{a\}, \{a, b\}\}$$

Let $A$ and $B$ be sets. A **relation** between $A$ and $B$ is any subset $R$ of $A \times B$.

$$aRb \iff (a, b) \in R$$

If $\mathcal{A}$ is a collection of sets, then $\mathcal{A}$ is called **pairwise disjoint** if

$$\forall A, B \in \mathcal{A}, \text{ where } A \neq B \text{ then } A \cap B = \emptyset$$

Let $A$ and $B$ be sets. $A \times B$ is the set of all ordered pairs $(a, b)$ such that $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

The **equivalence class** of $x \in S$ with respect to an equivalence relation $R$ is the set

$$E_x = \{y \in S : yRx\}$$

A relation $R$ on a set $S$ is an **equivalence relation** if for all $x, y, z \in S$ it satisfies the following criteria:

1. $xRx$ reflexivity
2. $xRy \Rightarrow yRx$ symmetry
3. $xRy$ and $yRz \Rightarrow xRz$ transitivity
Theorem

Definition

partition

function between A and B

Real Analysis I

Definition

domain

range & codomain

Real Analysis I

Definition

surjective or onto

injective or 1–1

Real Analysis I

Definition

bijective

characteristic or indicator function

Real Analysis I

Definition

image and pre-image

composition of functions

Real Analysis I
Let $A$ and $B$ be sets. A function between $A$ and $B$ is a nonempty relation $f \subseteq A \times B$ such that

$$[(a, b) \in f \text{ and } (a, b') \in f] \implies b = b'$$

A partition of a set $S$ is a collection $\mathcal{P}$ of nonempty subsets of $S$ such that

1. Each $x \in S$ belongs to some subset $A \in \mathcal{P}$.
2. For all $A, B \in \mathcal{P}$, if $A \neq B$, then $A \cap B = \emptyset$

A member of a set $\mathcal{P}$ is called a piece of the partition.

Let $A$ and $B$ be sets, and let $f \subseteq A \times B$ be a function between $A$ and $B$. The range of $f$ is the set of all second elements of members of $f$.

$$\text{rng } f = \{ b \in B : \exists a \in A \ni (a, b) \in f \}$$

The set $B$ is referred to as the codomain of $f$.

Let $A$ and $B$ be sets, and let $f \subseteq A \times B$ be a function between $A$ and $B$. The domain of $f$ is the set of all first elements of members of $f$.

$$\text{dom } f = \{ a \in A : \exists b \in B \ni (a, b) \in f \}$$

The function $f : A \to B$ is injective or (1–1) if:

$$\forall a, a' \in A, \quad f(a) = f(a') \implies a = a'$$

The function $f : A \to B$ is surjective or onto if $B = \text{rng } f$. Equivalently,

$$\forall b \in B, \quad \exists a \in A \ni b = f(a)$$

Let $A$ be a nonempty set and let $S \subseteq A$, then the characteristic function $\chi_S : A \to \{0, 1\}$ is defined by

$$\chi_S(a) = \begin{cases} 0 & a \notin S \\ 1 & a \in S \end{cases}$$

A function $f : A \to B$ is said to be bijective if $f$ is both surjective and injective.

Suppose $f : A \to B$ and $g : B \to C$, then the composition of $g$ with $f$ denoted by $g \circ f : A \to C$ is given by

$$(g \circ f)(x) = g(f(x))$$

In terms of ordered pairs this means

$$g \circ f = \{(a, c) \in A \times C : \exists b \in B \ni (a, b) \in f \land (b, c) \in g\}$$

Suppose $f : A \to B$, and $C \subseteq A$, then the image of $C$ under $f$ is

$$f(C) = \{ f(x) : x \in C \}$$

If $D \subseteq B$ then the pre-image of $D$ in $f$ is

$$f^{-1}(D) = \{ x \in A : f(x) \in D \}$$
Definition

inverse function

identity function

Real Analysis I

Definition
equinumerous

finite & infinite sets

Real Analysis I

Definition
cardinal number & transfinite
denumerable

Real Analysis I

Definition
countable & uncountable

power set

Real Analysis I

Definition

continuum hypothesis

algebraic & transcendental

Real Analysis I
A function that maps a set $A$ onto itself is called the **identity function** on $A$, and is denoted $i_A$.

If $f : A \to B$ is a bijection, then

$$
\begin{align*}
  f^{-1} \circ f &= i_A \\
  f \circ f^{-1} &= i_B
\end{align*}
$$

A set $S$ is said to be **finite** if $S = \emptyset$ or if there exists an $n \in \mathbb{N}$ and a bijection $f : \{1, 2, \ldots, n\} \to S$.

If a set is not finite, it is said to be **infinite**.

A set $S$ is said to be **denumerable** if there exists a bijection $f : \mathbb{N} \to S$.

Two sets $S$ and $T$ are **equinumerous**, denoted $S \sim T$, if there exists a bijection from $S$ onto $T$.

Let $I_n = \{1, 2, \ldots, n\}$. The **cardinal number** of $I_n$ is $n$. Let $S$ be a set. If $S \sim I_n$ then $S$ has $n$ elements.

The cardinal number of $\emptyset$ is defined to be 0.

Finally, if a cardinal number is not finite, it is said to be **transfinite**.

Given any set $S$, the **power set** of $S$ denoted by $\mathcal{P}(S)$ is the collection of all possible subsets of $S$.

If a set is finite or denumerable, then it is **countable**.

If a set is not countable, then it is **uncountable**.

A real number is said to be **algebraic** if it is a root of a polynomial with integer coefficients.

If a number is not algebraic, it is called **transcendental**.

Given that $|\mathbb{N}| = \aleph_0$ and $|\mathbb{R}| = c$, we know that $c > \aleph_0$, but is there any set with cardinality say $\lambda$ such that $\aleph_0 < \lambda < c$?

The conjecture that there is no such set was first made by Cantor and is known as the **continuum hypothesis**.
Axiom

well-ordering property of \( \mathbb{N} \)

Real Analysis I

Definition

basis for induction, induction step, induction hypothesis

Real Analysis I

Definition

recursion relation or recurrence relation

Real Analysis I

Axiom

field axioms

Real Analysis I

Definition

order axioms

Real Analysis I

Definition

absolute value

Real Analysis I

Theorem

triangle inequality

Real Analysis I

Definition

ordered field

Real Analysis I

Definition

irrational number

Real Analysis I

Definition

upper & lower bound

Real Analysis I
In the *Principle of Mathematical Induction*, part (1) which refers to \( P(1) \) being true is known as the **basis for induction**.

Part (2) where one must show that \( \forall k \in \mathbb{N}, P(k) \Rightarrow P(k+1) \) is known as the **induction step**.

Finally, the assumption in part (2) that \( P(k) \) is true is known as the **induction hypothesis**.

If \( S \) is a nonempty subset of \( \mathbb{N} \), then there exists an element \( m \in S \) such that \( \forall k \in S \ m \leq k \).

**A1 Closure under addition**
**A2 Addition is commutative**
**A3 Addition is associative**
**A4 Additive identity is 0**
**A5 Unique additive inverse of \( x \) is \(-x\)**

**M1 Closure under multiplication**
**M2 Multiplication is commutative**
**M3 Multiplication is associative**
**M4 Multiplicative identity is 1**
**M5 If \( x \neq 0 \), then the unique multiplicative inverse is \( \frac{1}{x} \)**

**DL** \( \forall x, y, z \in \mathbb{R}, x(y+z) = xy + xz \)

A **recurrence relation** is an equation that defines a sequence recursively: each term of the sequence is defined as a function of the preceding terms.

The Fibonacci numbers are defined using the linear recurrence relation:

\[
F_n = F_{n-2} + F_{n-1}
\]

\[
F_1 = 1
\]

\[
F_2 = 1
\]

**O1** \( \forall x, y \in \mathbb{R} \) exactly one of the relations \( x = y, x < y, x > y \) holds. (trichotomy)

**O2** \( \forall x, y, z \in \mathbb{R}, x < y \text{ and } y < z \Rightarrow x < z \). (transitivity)

**O3** \( \forall x, y, z \in \mathbb{R}, x < y \Rightarrow x + z < y + z \)

**O4** \( \forall x, y, z \in \mathbb{R}, x < y \text{ and } z > 0 \Rightarrow xz < yz \).

Let \( x, y \in \mathbb{R} \) then

\[
|x + y| \leq |x| + |y|
\]

Alternatively,

\[
|a - b| \leq |a - c| + |c - b|
\]

Let \( S \) be a subset of \( \mathbb{R} \). If there exists an \( m \in \mathbb{R} \) such that \( m \geq s \ \forall \ s \in S \), then \( m \) is called an **upper bound** of \( S \).

Similarly, if \( m \leq s \ \forall \ s \in S \), then \( m \) is called a **lower bound** of \( S \).

Suppose \( x \in \mathbb{R} \). If \( x \neq \frac{m}{n} \) for some \( m, n \in \mathbb{Z} \), then \( x \) is **irrational**.
Axiom

Completeness Axiom

Archimedean ordered field

Definition

bounded

maximum & minimum

supremum

infimum

definite & radius

dense

extended real numbers

neighborhood & radius

deleted neighborhood
If \( m \) is an upper bound of \( S \) and also in \( S \), then \( m \) is called the **maximum** of \( S \).

Similarly, if \( m \) is a lower bound of \( S \) and also in \( S \), then \( m \) is called the **minimum** of \( S \).

A set \( S \) is said to be **bounded** if it is bounded above and bounded below.

Let \( S \) be a nonempty subset of \( \mathbb{R} \). If \( S \) is bounded below, then the **greatest lower bound** is called the **infimum**, and is denoted \( \inf S \).

\[
m = \inf S \iff 
    \begin{align*}
    (a) & \quad m \leq s, \quad \forall s \in S \text{ and} \\
    (b) & \quad \text{if } m' > m, \text{ then } \exists s' \in S \ni s' < m'
    \end{align*}
\]

Let \( S \) be a nonempty subset of \( \mathbb{R} \). If \( S \) is bounded above, then the **least upper bound** is called the **supremum**, and is denoted \( \sup S \).

\[
m = \sup S \iff 
    \begin{align*}
    (a) & \quad m \geq s, \quad \forall s \in S \text{ and} \\
    (b) & \quad \text{if } m' < m, \text{ then } \exists s' \in S \ni s' > m'
    \end{align*}
\]

An ordered field \( F \) has the **Archimedean property** if

\[
\forall x \in F \quad \exists n \in \mathbb{N} \ni x < n
\]

Every nonempty subset \( S \) of \( \mathbb{R} \) that is bounded above has a least upper bound. That is, \( \sup S \) exists and is a real number.

For convenience, we extend the set of real numbers with two symbols \( \infty \) and \( -\infty \), that is \( \mathbb{R} \cup \{\infty, -\infty\} \).

Then for example if a set \( S \) is not bounded above, then we can write

\[
\sup S = \infty
\]

A set \( S \) is **dense** in a set \( T \) if

\[
\forall t_1, t_2 \in T \quad \exists s \in S \ni t_1 < s < t_2
\]

Let \( x \in \mathbb{R} \) and \( \varepsilon > 0 \), then a **deleted neighborhood** of \( x \) is

\[
N^*(x; \varepsilon) = \{y \in \mathbb{R} : 0 < |y - x| < \varepsilon\}
\]

Let \( x \in \mathbb{R} \) and \( \varepsilon > 0 \), then a **neighborhood** of \( x \) is

\[
N(x; \varepsilon) = \{y \in \mathbb{R} : |y - x| < \varepsilon\}
\]

The number \( \varepsilon \) is referred to as the **radius** of \( N(x; \varepsilon) \).
Definition

interior point

boundary point

Real Analysis I

Definition

closed and open sets

accumulation point

Real Analysis I

Definition

isolated point

closure of a set

Real Analysis I

Definition

open cover

subcover

Real Analysis I

Definition

compact set

sequence

Real Analysis I
A point \( x \in \mathbb{R} \) is a **boundary point** of \( S \) if
\[
\forall \varepsilon > 0, \quad N(x; \varepsilon) \cap S \neq \emptyset \quad \text{and} \quad N(x; \varepsilon) \cap (\mathbb{R} \setminus S) \neq \emptyset
\]
In other words, every neighborhood of a boundary point must intersect the set \( S \) and the complement of \( S \) in \( \mathbb{R} \).
The set of all boundary points of \( S \) is denoted \( \text{bd} \, S \).

Suppose \( S \subseteq \mathbb{R} \), then a point \( x \in \mathbb{R} \) is called an **accumulation point** of \( S \) if
\[
\forall \varepsilon > 0, \quad N^*(x; \varepsilon) \cap S \neq \emptyset
\]
In other words, every deleted neighborhood of \( x \) contains a point in \( S \).
The set of all accumulation points of \( S \) is denoted \( S' \).

Let \( S \subseteq \mathbb{R} \). The **closure** of \( S \) is defined by
\[
\text{cl} \, S = S \cup S'
\]
In other words, the closure of a set is the set itself unioned with its set of accumulation points.

Suppose \( \mathcal{G} \subseteq \mathcal{F} \) are both families of indexed sets that cover a set \( S \), then since \( \mathcal{G} \) is a subset of \( \mathcal{F} \) it is called a **subcover** of \( S \).

Let \( S \subseteq \mathbb{R} \) and \( x \in S \) and \( x \not\in S' \), then \( x \) is called an **isolated point** of \( S \).

An **open cover** of a set \( S \) is a family or collection of sets whose union contains \( S \).
\[
S \subseteq \mathcal{F} = \{ F_n : n \in \mathbb{N} \}
\]

A set \( S \) is **compact** iff every open cover of \( S \) contains a finite subcover of \( S \).

Note: This is a difficult definition to use because to show that a set is compact you must show that every open cover contains a finite subcover.
Definition

converge & diverge

Definition

bounded sequence

Real Analysis I

Definition

diverge to $+\infty$

Definition

diverge to $-\infty$

Real Analysis I

Definition

nondecreasing, nonincreasing & monotone

Definition

increasing & decreasing

Real Analysis I

Definition

Cauchy sequence

Definition

subsequence

Real Analysis I

Definition

subsequential limit

Definition

lim sup & lim inf

Real Analysis I
A sequence is said to be **bounded** if its range \( \{s_n : n \in \mathbb{N}\} \) is bounded. Equivalently if,

\[
\exists M \geq 0 \text{ such that } \forall n \in \mathbb{N}, \ |s_n| \leq M
\]

A sequence \((s_n)\) is said to **converge** to \(s \in \mathbb{R}\), denoted \((s_n) \rightarrow s\) if

\[
\forall \varepsilon > 0, \ \exists N \text{ such that } \forall n \in \mathbb{N}, \ n > N \Rightarrow |s_n - s| < \varepsilon
\]

If a sequence does not converge, it is said to **diverge**.

A sequence \((s_n)\) is said to diverge to \(-\infty\) if

\[
\forall M \in \mathbb{R}, \ \exists N \text{ such that } \ n > N \Rightarrow s_n < M
\]

A sequence \((s_n)\) is said to diverge to \(+\infty\) if

\[
\forall M \in \mathbb{R}, \ \exists N \text{ such that } \ n > N \Rightarrow s_n > M
\]

A sequence \((s_n)\) is **increasing** if

\[
s_n < s_{n+1} \ \forall n \in \mathbb{N}
\]

A sequence \((s_n)\) is **decreasing** if

\[
s_n > s_{n+1} \ \forall n \in \mathbb{N}
\]

A sequence \((s_n)\) is **nondecreasing** if

\[
s_n \leq s_{n+1} \ \forall n \in \mathbb{N}
\]

A sequence \((s_n)\) is **nonincreasing** if

\[
s_n \geq s_{n+1} \ \forall n \in \mathbb{N}
\]

A sequence is **monotone** if it is either nondecreasing or nonincreasing.

A sequence \((s_n)\) is said to be a **Cauchy sequence** if

\[
\forall \varepsilon > 0, \ \exists N \text{ such that } \forall m, n > N \Rightarrow |s_n - s_m| < \varepsilon
\]

Suppose \(S\) is the set of all subsequential limits of a sequence \((s_n)\). The \(\lim \sup (s_n)\), shorthand for the limit superior of \((s_n)\) is defined to be

\[
\lim \sup (s_n) = \sup S
\]

The \(\lim \inf (s_n)\), shorthand for the limit inferior of \((s_n)\) is defined to be

\[
\lim \inf (s_n) = \inf S
\]

A **subsequential limit** of a sequence \((s_n)\) is the limit of some subsequence of \((s_n)\).
oscillating sequence

limit of a function

sum, product, multiple, & quotient of functions

right–hand limit

left–hand limit

continuous function at a point

continuous on $S$

bounded function

uniform continuity

extension of a function
Suppose \( f : D \rightarrow \mathbb{R} \) where \( D \subseteq \mathbb{R} \), and suppose \( c \) is an accumulation point of \( D \). Then the **limit of \( f \) at \( c \)** is \( L \) is denoted by

\[
\lim_{x \to c} f(x) = L
\]

and defined by

\[
\forall \varepsilon > 0, \ \exists \delta > 0 \text{ such that } \ |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon
\]

Let \( f : (a, b) \rightarrow \mathbb{R} \), then the **right–hand limit** of \( f \) at \( a \) is denoted

\[
\lim_{x \to a^+} f(x) = L
\]

and defined by

\[
\forall \varepsilon > 0, \ \exists \delta > 0 \text{ such that } \ a < x < a + \delta \Rightarrow |f(x) - L| < \varepsilon
\]

Let \( f : D \rightarrow \mathbb{R} \) where \( D \subseteq \mathbb{R} \), and suppose \( c \in D \), then \( f \) is **continuous** at \( c \) if

\[
\forall \varepsilon > 0, \ \exists \delta > 0 \text{ such that } \ |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon
\]

A function is said to be **bounded** if its range is bounded. Equivalently, \( f : D \rightarrow \mathbb{R} \) is bounded if

\[
\exists M \in \mathbb{R} \text{ such that } \forall x \in D, \ |f(x)| \leq M
\]

Suppose \( f : (a, b) \rightarrow \mathbb{R} \), then the **extension of \( f \)** is denoted \( \tilde{f} : [a, b] \rightarrow \mathbb{R} \) and defined by

\[
\tilde{f}(x) = \begin{cases} 
  u & x = a \\
  f(x) & a < x < b \\
  v & x = b
\end{cases}
\]

where \( \lim_{x \to a} f(x) = u \) and \( \lim_{x \to b} f(x) = v \).

If \( \lim \inf (s_n) < \lim \sup (s_n) \), then we say that the sequence \( (s_n) \) **oscillates**.

Let \( f : D \rightarrow \mathbb{R} \) and \( g : D \rightarrow \mathbb{R} \), then we define:

1. **sum** \( (f + g)(x) = f(x) + g(x) \)
2. **product** \( (fg)(x) = f(x)g(x) \)
3. **multiple** \( (kf)(x) = kf(x) \quad k \in \mathbb{R} \)
4. **quotient** \( \left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)} \) if \( g(x) \neq 0 \ \forall x \in D \)

Let \( f : (a, b) \rightarrow \mathbb{R} \), then the **left–hand limit** of \( f \) at \( b \) is denoted

\[
\lim_{x \to b^-} f(x) = L
\]

and defined by

\[
\forall \varepsilon > 0, \ \exists \delta > 0 \text{ such that } \ b - \delta < x < b \Rightarrow |f(x) - L| < \varepsilon
\]

Let \( f : (a, b) \rightarrow \mathbb{R} \), then the **extension of \( f \)** is denoted \( \tilde{f} : [a, b] \rightarrow \mathbb{R} \) and defined by

\[
\tilde{f}(x) = \begin{cases} 
  u & x = a \\
  f(x) & a < x < b \\
  v & x = b
\end{cases}
\]

where \( \lim_{x \to a} f(x) = u \) and \( \lim_{x \to b} f(x) = v \).

A function \( f : D \rightarrow \mathbb{R} \) is **uniformly continuous on** \( D \) if

\[
\forall \varepsilon > 0, \ \exists \delta > 0 \text{ such that } \ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon
\]
Definition

differentiable at a point

strictly increasing function

strictly decreasing function

Definition

limit at \( \infty \)
tends to \( \infty \)

Definition

Taylor polynomials for \( f \) at \( x_0 \)

Taylor series

Definition

partition of an interval
refinement of a partition

upper sum

Definition

lower sum

upper integral

lower integral
A function \( f : D \to \mathbb{R} \) is said to be **strictly increasing** if
\[
\forall x_1, x_2 \in D, \quad x_1 < x_2 \Rightarrow f(x_1) < f(x_2)
\]
A function \( f : D \to \mathbb{R} \) is said to be **strictly decreasing** if
\[
\forall x_1, x_2 \in D, \quad x_1 < x_2 \Rightarrow f(x_1) > f(x_2)
\]

Suppose \( f : (a, \infty) \to \mathbb{R} \), then we say \( f \) **tends to** \( \infty \) as \( x \to \infty \) and denote it by
\[
\lim_{x \to \infty} f(x) = \infty
\]
iff
\[
\forall M \in \mathbb{R}, \quad \exists N > a \text{ such that } x > N \Rightarrow f(x) > M
\]

If \( f \) has derivatives of all orders in a neighborhood of \( x_0 \), then the limit of the Taylor polynomials is an infinite series called the **Taylor series** of \( f \) at \( x_0 \).
\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n
\]
\[
= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots
\]

Suppose \( f \) is a bounded function on \([a, b]\) and \( P = \{x_0, \ldots, x_n\} \) is a partition of \([a, b]\).
For each \( i \in \{1, \ldots, n\} \) let
\[
M_i(f) = \sup\{f(x) : x \in [x_{i-1}, x_i]\}.
\]
We define the **upper sum** of \( f \) with respect to \( P \) to be
\[
U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i
\]
where \( \Delta x_i = x_i - x_{i-1} \).

Suppose \( f \) is a bounded function on \([a, b]\). We define the **upper integral** of \( f \) on \([a, b]\) to be
\[
U(f) = \inf\{U(f, P) : P \text{ any partition of } [a, b]\}.
\]
Similarly, we define the **lower integral** of \( f \) on \([a, b]\) to be
\[
L(f) = \sup\{L(f, P) : P \text{ any partition of } [a, b]\}.
\]
Suppose \( f : I \to \mathbb{R} \) where \( I \) is an interval containing the point \( c \). Then \( f \) is **differentiable at** \( c \) if the limit
\[
\lim_{x \to c} \frac{f(x) - f(c)}{x - c}
\]
exists and is finite. Whenever this limit exists and is finite, we denote the **derivative of** \( f \) **at** \( c \) by
\[
f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}
\]

Suppose \( f : (a, \infty) \to \mathbb{R} \), then the **limit at infinity** of \( f \) denoted
\[
\lim_{x \to \infty} f(x) = L
\]
iff
\[
\forall \varepsilon > 0, \quad \exists N > a \text{ such that } x > N \Rightarrow |f(x) - L| < \varepsilon
\]

A partition of an interval \([a, b]\) is a finite set of points \( P = \{x_0, x_1, x_2, \ldots, x_n\} \) such that
\[
a = x_0 < x_1 < \ldots < x_n = b
\]
If \( P \) and \( P' \) are two partitions of \([a, b]\) where \( P \subset P' \) then \( P' \) is called a **refinement** of \( P \).

Suppose \( f \) is a bounded function on \([a, b]\) and \( P = \{x_0, x_1, x_2, \ldots, x_n\} \) is a partition of \([a, b]\).
For each \( i \in \{1, \ldots, n\} \) let
\[
m_i(f) = \inf\{f(x) : x \in [x_{i-1}, x_i]\}.
\]
We define the **lower sum** of \( f \) with respect to \( P \) to be
\[
L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i
\]
where \( \Delta x_i = x_i - x_{i-1} \).
Definition

Definition

Riemann integrable

monotone function

Real Analysis I

Real Analysis I

Definition

Definition

proper integral

improper integral

Real Analysis I

Real Analysis I

Definition

Definition

integral convergence

infinite series

integral divergence

partial sum

Real Analysis I

Real Analysis I

Definition

Definition

convergent series

divergent series

sum

diverge to $+\infty$

Real Analysis I

Real Analysis I

Definition

Definition

harmonic series

geometric series

Real Analysis I

Real Analysis I
A function is said to be **monotone** if it is either increasing or decreasing.

A function is increasing if \( x < y \Rightarrow f(x) \leq f(y) \).
A function is decreasing if \( x < y \Rightarrow f(x) \geq f(y) \).

An **improper integral** is the limit of a definite integral, as an endpoint of the interval of integration approaches either a specified real number or \( \infty \) or \(-\infty\) or, in some cases, as both endpoints approach limits.

Let \( f : (a, b] \to \mathbb{R} \) be integrable on \([c, b] \forall c \in (a, b]\). If \( \lim_{c \to a^+} \int_c^b f \) exists then

\[
\int_a^b f = \lim_{c \to a^+} \int_c^b f
\]

Let \((a_k)\) be a sequence of real numbers, then we can create a new sequence of numbers \((s_n)\) where each \(s_n\) in \((s_n)\) corresponds to the sum of the first \(n\) terms of \((a_k)\).
This new sequence of sums is called an **infinite series** and is denoted by \( \sum_{n=0}^{\infty} a_n \).

The \(n\)-th **partial sum** of the series, denoted by \(s_n\) is defined to be

\[
s_n = \sum_{k=0}^{n} a_k
\]

If a series does not converge then it is **divergent**.

If the \( \lim_{n \to \infty} s_n = +\infty \) then the series is said to **diverge to** \( +\infty \).

The **geometric series** is given by

\[
\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots
\]

The geometric series converges to \( \frac{1}{1-x} \) for \( |x| < 1 \), and diverges otherwise.

Let \( f : [a, b] \to \mathbb{R} \) be a bounded function. If \( L(f) = U(f) \), then we say \( f \) is **Riemann integrable** or **just integrable**. Furthermore,

\[
\int_a^b f = \int_a^b f(x)dx = L(f) = U(f)
\]

is called the **Riemann integral** or just the **integral** of \( f \) on \([a, b]\).

When a function \( f \) is bounded and the interval over which it is integrated is bounded, then if the integral exists it is called a **proper integral**.

Suppose \( f : (a, b] \to \mathbb{R} \) is integrable on \([c, b] \forall c \in (a, b]\), furthermore let \( L = \lim_{c \to a^+} \int_c^b f \). If \( L \) is finite, then the improper integral \( \int_a^b f \) is said to **converge** to \( L \).

If \( L = \infty \) or \( L = -\infty \), then the improper integral is said to **diverge**.

If \((s_n)\) converges to a real number say \(s\), then we say that the series \( \sum_{n=0}^{\infty} a_n = s \) is **convergent**.

Furthermore, we call \( s \) the **sum** of the series.

The **harmonic series** is given by

\[
\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots
\]

The harmonic series diverges to \(+\infty\).
**Definition**

- converge absolutely
- converge conditionally
- power series

**Real Analysis I**

**Definition**

- radius of convergence
- interval of convergence

**Real Analysis I**

**Definition**

- converges pointwise
- converges uniformly

**Real Analysis I**
Given a sequence \((a_n)\) of real numbers, then the series
\[
\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots
\]
is called a **power series**. The number \(a_n\) is called the
\(n\)th coefficient of the series.

If \(\sum |a_n|\) converges then the series \(\sum a_n\) is said to
**converge absolutely**.

If \(\sum a_n\) converges, but \(\sum |a_n|\) diverges, then the series
\(\sum a_n\) is said to **converge conditionally**.

The **interval of convergence** of a power series is the
set of all \(x \in \mathbb{R}\) such that \(\sum_{n=0}^{\infty} a_n x^n\) converges.

By theorem we see that (for a power series centered at 0) this set will either be \(\{0\}\), \(\mathbb{R}\) or a bounded interval
centered at 0.

Let \((f_n)\) be a sequence of functions defined on a subset
\(S\) of \(\mathbb{R}\). Then \((f_n)\) **converges uniformly** on \(S\) to a
function \(f\) defined on \(S\) if
\[
\forall \varepsilon > 0, \quad \exists N \text{ such that } \forall x \in S \quad n > N \Rightarrow |f_n(x) - f(x)| < \varepsilon
\]

Let \((f_n)\) be a sequence of functions defined on a subset
\(S\) of \(\mathbb{R}\). Then \((f_n)\) **converges pointwise** on \(S\) if for
each \(x \in S\) the sequence of numbers \((f_n(x))\) converges.
If \((f_n)\) converges pointwise on \(S\), then we define \(f : S \to \mathbb{R}\) by
\[
f(x) = \lim_{n \to \infty} f_n(x)
\]
for each \(x \in S\), and we say that \((f_n)\) converges to \(f\)
pointwise on \(S\).