1. Consider the forced, linear system,

\[ y^{(4)} + 2y^{(3)} + y'' = 6x. \] (1)

(a) Find the homogeneous solution \( y_h \).

(b) Find a particular solution \( y_p \).

*Hint:* Your trial particular solution should be \( y_p = Ax^2 + Bx^3 \).

(c) Find the general solution \( y(x) = y_h + y_p \).
2. Consider the following series RLC circuit:

Using Kirchoff’s law for closed loops yields the following differential equation which governs the charge on the capacitor, \( q(t) \).

\[
L q'' + R q' + \frac{1}{C} q = V_0 \cos \omega t \\
\tag{2}
\]

Assume initial conditions: \( q(0) = 0 \), \( q'(0) = 0 \), and the following values for \( L \), \( R \) and \( C \):

\[
L = 1 \text{ H (Henry)}, \\
R = 0 \text{ Ω (Ohm)}, \\
C = \frac{25}{9} \text{ F (Farad)}, \\
V_0 = 3 \text{ V (Volt)}. 
\]

(a) What is the natural angular frequency, \( \omega_0 \) of this system?  
*Hint:* Recall \( \omega_0 \) is the angular frequency of the solution to the unforced (associated homogeneous) equation:

\[
L q'' + R q' + \frac{1}{C} q = 0. 
\]

(b) Assume \( \omega \neq \omega_0 \). Use the method of undetermined coefficients to solve for a particular solution, \( q_p \), of equation (2). Finally, use the general solution \( q(t) = q_h + q_p \) to solve the IVP.

(c) Write down the specific case of the solution \( q(t) \) (found in part b) for \( \omega = 0.5 \). Compute the period \( (T = \frac{2\pi}{\omega}) \) of this solution, which is a superposition of two cosine functions. Use MATLAB or Maple to graph one period of the solution. What phenomenon is exhibited by this solution?

(d) Now let \( \omega = \omega_0 \). Use the method of undetermined coefficients to solve for a new particular solution \( q_p \). Then use \( q(t) = q_h + q_p \) to solve the initial value problem. Graph this solution over the interval \( 0 \leq t \leq 60 \) seconds. What phenomenon does this solution exhibit?
3. Consider the same RLC circuit as in problem 2. For this problem, take $L = 1$ H, $R = 1.2$ Ω, $C^{-1} = 0.36$ V · C$^{-1}$, $V_0 = 3$ V, and $\omega = 1$. This gives us the differential equation:

$$q'' + 1.2q' + 0.36q = 3 \cos t.$$  \hspace{1cm} (3)

(a) Use the method of undetermined coefficients to find a particular solution $q_p(t)$ to this differential equation.

(b) Use the particular solution $q_p$ found in part (a) and the solution $q_h$ to the corresponding homogeneous equation to write down the general solution to this differential equation. Identify the “steady periodic” and “transient” parts of this general solution.

(c) Given the initial conditions $q(0) = 0$, $q'(0) = 0$, solve the resulting IVP for $q(t)$ satisfying the differential equation at the beginning of this problem (you may use Maple).

(d) Graph, on a single plot, the solution to the IVP in part (c) as well as the steady periodic solution identified in part (b). Choose a time interval so that you can clearly see the convergence of the IVP solution to the steady periodic solution.
4. The *Laplace transform* is an operator \( \mathcal{L} \), which takes as input a function of time, \( f(t) \) and outputs a function of frequency, \( F(s) \) according to the rule:

\[
\mathcal{L} \{ f(t) \} = F(s) = \int_0^{\infty} e^{-st} f(t) \, dt
\]

(4)

Transform methods are very useful for solving differential equations, but a physical relationship between \( F(s) \) and \( f(t) \) is not immediately clear. The goal of this problem is to give intuition for the variable \( s \).

(a) Consider the following functions:

\[
\begin{align*}
  f_1(t) &= \begin{cases} 
    -1 & 0 \leq t \leq 1 \\
    +1 & 1 < t 
  \end{cases} \\
  f_2(t) &= \cos 2t \\
  f_3(t) &= \sin 3t
\end{align*}
\]

Find the Laplace transforms \( F_1(s) \), \( F_2(s) \), \( F_3(s) \) of these functions. It should be straightforward to compute \( F_1(s) \) via hand by breaking up the integral into two integrals, one with bounds 0 and 1, and the other with bounds 1 and \( \infty \). You can just look up \( F_2(s) \) and \( F_3(s) \) using the table in your textbook.

(b) Rewrite each of the Laplace transforms found in part (a), by replacing the variable \( s \) with \( i\omega \). Here \( i \) is the imaginary number \( i^2 = -1 \), and \( \omega \) is a real number which represents an angular frequency.

(c) If \( z = x + iy \) is a complex number, its *complex conjugate* is \( z^* = x - iy \). The non-negative real number \( |z| = \sqrt{zz^*} \) is called the *magnitude* of \( z \). Find the magnitude of \( F_1(i\omega) \), \( F_2(i\omega) \), and \( F_3(i\omega) \). *Hint:* For this problem, the complex conjugate of \( F(i\omega) \) is \( F(-i\omega) \), and \( e^{\pm ix} = \cos x \pm i \sin x \).

(d) Plot the magnitudes \( |F_1(i\omega)| \), \( |F_2(i\omega)| \), and \( |F_3(i\omega)| \). Can you relate these plots to the frequency of \( f_2(t) \) and \( f_3(t) \)? Based on that relationship, what can you say about the frequencies of \( f_1(t) \)?