1. Let $M_n$ be the $n \times n$ matrix with all 2's along “the other diagonal”, and 0's everywhere else. For example,

\[
M_4 = \begin{bmatrix}
0 & 0 & 0 & 2 \\
0 & 0 & 2 & 0 \\
0 & 2 & 0 & 0 \\
2 & 0 & 0 & 0
\end{bmatrix}.
\]

Let $d_n = \det(M_n)$.

(1) Find a formula expressing $d_n$ in terms of $d_{n-1}$ for positive integers $n \geq 2$.

(2) Find $d_1, d_2, \ldots, d_8$. Do you see a pattern? Find a closed formula for $d_n$ and justify your formula by the mathematical induction.

(3) Find $d_{100}$.

2. Let $A = \begin{bmatrix}
\vec{v}_1 \\
\vec{v}_2 \\
\vec{v}_3
\end{bmatrix}$ be the $3 \times 3$ matrix with rows $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

Suppose $\det(A) = 3$. Find the following.

(1) $\det\left(\begin{bmatrix} -\vec{v}_2 \\
-2\vec{v}_1 \\
\vec{v}_3 \end{bmatrix}\right)$ =

(2) $\det\left(\begin{bmatrix} -\vec{v}_2 \\
-2\vec{v}_3 \\
\vec{v}_1 \end{bmatrix}\right)$ =

(3) $\det\left(\begin{bmatrix} -2\vec{v}_2 + 3\vec{v}_3 \\
-2\vec{v}_1 \\
6\vec{v}_2 - 9\vec{v}_3 \end{bmatrix}\right)$ =

3. Consider the parallelepiped $V$ in $\mathbb{R}^3$ defined by three vectors

\[
\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}.
\]

in $\mathbb{R}^3$.

(1) Find the volume of the parallelepiped $V$.

(2) What is the volume of the image $T(V)$ of $V$ under the linear transformation $T(\vec{x}) = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 0 \\ 4 & 5 & 6 \end{bmatrix} \vec{x}$.
4. Consider the linear transformation \( T : P_2 \to P_2 \) defined by two cases:

(1) \( T(f(x)) = f(2x - 1) \)
(2) \( T(f(x)) = f'(x) \)

Let \( B = (1, x, x^2) \) be the standard basis of \( P_2 \). Answer the following questions for each of maps above.

(a) Find the \( B \)-matrix \( A \) of \( T \)
(b) Find the determinant of \( T \).
(c) Find all eigenvalues and all eigenvectors of \( T \). Specify the algebraic multiplicity and geometric multiplicity of each eigenvalue.
(d) Determine whether \( P_2 \) has an eigenbasis of \( T \). If \( P_2 \) has an eigenbasis of \( T \), let \( D \) be an eigenbasis and find the \( D \)-matrix \( B \) of \( T \).

\(<\text{True or False questions}\>)

Determine whether the following statement is True or False.

(1) If 0 is an eigenvalue of a matrix \( A \), then \( \det(A) = 0 \).
(2) If \( \vec{v} \) is an eigenvector of \( A \), then \( \vec{v} \) must be an eigenvector of \( A^3 \) as well.
(3) The matrix of any orthogonal projection on a line \( L \) in \( \mathbb{R}^2 \) gives an eigenbasis of \( \mathbb{R}^2 \).
(4) If an invertible matrix \( A \) gives an eigenbasis of \( \mathbb{R}^n \), then \( A^{-1} \) must give an eigenbasis of \( \mathbb{R}^n \) as well.
(5) If \( \vec{v} \) and \( \vec{w} \) are linearly independent eigenvectors of \( A \), then \( \vec{v} + \vec{w} \) is also an eigenvector of \( A \).
(6) \( \det(AB) = \det(A)\det(B) \).
(7) \( \det(A + B) = \det(A) + \det(B) \).
(8) \( \det(AB) = \det(BA) \).
(9) If all the entries of a 7\( \times \)7 matrix \( A \) are 7, then \( \det(A) \) must be \( 7^7 \).
(10) If the determinant of an 5\( \times \)5 matrix \( A \) is 5, then its rank must be 5.
(11) If \( A \) is any symmetric matrix, then \( \det(A) = 1 \) or \( -1 \).
(12) If \( A \) is any skew-symmetric 4\( \times \)4 matrix, then \( \det(A) = 0 \).
(13) If \( A \) is any skew-symmetric 5\( \times \)5 matrix, then \( \det(A) = 0 \).
(14) If \( A \) is orthogonal, then \( \det(A) = 1 \) or \( -1 \).
(15) There exists invertible 3\( \times \)3 matrix \( A \) and \( S \) such that \( S^{-1}AS = -A \).
(16) There exists a 3\( \times \)3 matrix \( A \) such that \( A^2 = -I_3 \).
(17) If an \( n \times n \) matrix \( A \) is invertible, then \( \text{adj}(A) \) is invertible as well.
(18) If \( A \) is a 6\( \times \)6 matrix with three distinct eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \) and \( E_{\lambda_1} \) has geometric multiplicity 2 and \( E_{\lambda_2} \) has geometric multiplicity 3. Then \( \mathbb{R}^6 \) has an eigenbasis of \( A \).

Good-luck!