Exponential Functions

An exponential function grows (or decays) by the same relative amount (the same percentage) in each unit of time. For a quantity $Q$ growing exponentially with a fractional growth rate $r$,

$$Q = Q_0 \times (1+r)^t$$

where

- $Q = \text{value of exponentially growing quantity after time } t$
- $Q_0 = \text{initial value of the quantity}$
- $r = \text{fractional growth rate of the quantity (this is the percentage by which the quantity is growing, converted to decimal)}$
- $t = \text{time}$

Negative values of $r$ correspond to exponential decay. Note that the units of time used for $t$ and $r$ must be the same. For example, if the fractional growth rate is 0.05 per month, then $t$ must also be in months.

Example 1:
The population of Greensville is increasing at a rate of 5.6% per year. If the population today is 8,000, what will it be 10 years from now?

Solution:
Using our formula above with $Q_0 = 8,000$, $r = .056$ (per year), and $t = 10$ (years), we have

$$Q = Q_0 \times (1+r)^t$$

$$Q = 8,000 \times (1+.056)^{10}$$

$$Q = 13,795$$

This tells us that 10 years from now the population will be 13,795.
Example 2:
You have an investment that is increasing at a rate of 2\% per month. If you have $700 invested today, how much will you have two years from now? When will you have $10,000?

Solution:
We have $Q_0 = 700$, $r = 0.02$ (per month), and $t = 2$ (years). We can’t use our formula until we have the same units for $r$ and $t$. 2 years is the same as 48 months, so we can list our variables as $Q_0 = 700$, $r = 0.02$ (per month), and $t = 48$ (months), and now we are ready to plug these values into our formula.

\[
Q = Q_0 \times (1+r)^t
\]

\[
Q = 700 \times (1+0.02)^{48}
\]

\[
Q = 1,810.95
\]

This tells us that 2 years from now we will have $1,810.95.

When will we have $10,000?

In order to answer this question, we need to solve our formula for $t$. We can do this using logs, and we have

\[
t = \frac{\log \left( \frac{Q}{Q_0} \right)}{\log (1 + r)}
\]

\[
t = \frac{\log \left( \frac{10,000}{700} \right)}{\log (1 + 0.02)}
\]

\[
t = 134.29
\]

Since $r$ was in terms of months, this tells us that in 134.29 months, or about 11 years, we will have $10,000.
Example 3: Suppose that you take 800 mg of aspirin at 1pm. 5 hours later, you have 100 mg left in your bloodstream. What is the rate of decrease of the aspirin in your bloodstream?

Solution: In this problem, we have $Q = 100$ mg, $Q_0 = 800$ mg, and $t = 5$ (hours). We want to find $r$. We can solve our equation for $r$ as follows

$$r = \left( \frac{Q}{Q_0} \right)^{\frac{1}{t}} - 1$$

Plugging in our values for $Q$, $Q_0$, and $t$, we have

$$r = -0.34$$

This tells us that the aspirin is decreasing at a rate of 34% each hour.

Exercises

1.) A population of rabbits is increasing at a rate of 1.5% per month. If there are 60 rabbits today, how many will there be after 10 months?

2.) Your antique watch is increasing in value at a rate of 5% each year. If it is worth $500 today, how much will it be worth 3 years from now?
3.) The population in Ghettosburg was 800 6 years ago, and is 400 today. What is the rate of decrease? (Hint: follow Example 3)

4.) Your savings account is growing at a rate of 3.2% per month. How long will it take for your account to triple in size? (Hint: Pick any value you like for \( Q_0 \), and set \( Q = 3Q_0 \), for example you could let \( Q_0 = $100 \), and \( Q = $300 \), then follow Example 2)