PROBLEM 1

Consider the function
\[ f(x, y) = y\sqrt{x} - y^2 - 5x + 19y, \quad x > 0. \]
Find and classify the critical point of the function.

Find the first derivative and set it equal to zero.
\[ f(x, y) = y\sqrt{x} - y^2 - 5x + 19y \]
\[ f_x(x, y) = \frac{1}{2}yx^{\frac{1}{2}} - 5 \quad f_y(x, y) = x^{\frac{1}{2}} - 2y + 19 \]
\[ f_x = \frac{1}{2}yx^{\frac{1}{2}} - 5 = 0 \quad f_y = x^{\frac{1}{2}} - 2y + 19 = 0 \quad (5 \text{ pts}) \]

Find the critical points
Solve \( f_x \) for \( y \)
\[ \frac{1}{2}yx^{\frac{1}{2}} - 5 = 0 \quad \Rightarrow \quad y = 10\sqrt{x} \]
Substitute \( 10\sqrt{x} \) in for \( y \) in \( f_y \)
\[ \sqrt{x} - 2(10\sqrt{x}) + 19 = 0 \]
\[ -19\sqrt{x} + 19 = 0 \quad \Rightarrow \quad x = 1 \]
\[ y = 10\sqrt{1} \quad \Rightarrow \quad y = 10 \]
\[ x = 1 \quad y = 10 \quad (5 \text{ pts}) \]

There are two methods to classify this critical point. Choose one.

Method 1 - Find the Determinant
\[
\begin{bmatrix}
    f_{xx} & f_{xy} \\
    f_{yx} & f_{yy}
\end{bmatrix}
= \begin{bmatrix}
    -\frac{1}{4}yx^{\frac{1}{2}} & \frac{1}{2}x^{\frac{7}{2}} \\
    \frac{1}{2}x^{\frac{1}{2}} & -2
\end{bmatrix}
\]
\[
\begin{bmatrix}
    \frac{1}{4}yx^{\frac{1}{2}} & \frac{1}{2}x^{\frac{7}{2}} \\
    \frac{1}{2}x^{\frac{1}{2}} & -2
\end{bmatrix}
\bigg|_{y=10}^{x=1}
= \begin{bmatrix}
    -\frac{10}{4} & \frac{1}{2} \\
    \frac{1}{2} & -2
\end{bmatrix}
\quad (5 \text{ pts})
\]

The determinant is greater than zero. \( 5 - \frac{1}{4} > 0 \) and
\[ f_{xx} \] is less than zero therefore it is a local maximum. \( (5 \text{ pts}) \)
Method 2 - Taylor Polynomial

\[ p_2(x,y) = f(1,10) + \frac{1}{2} \left[ -\frac{10}{4} (x-1)^2 + (x-1)(y-10) + 2(y-10)^2 \right] \]

\[ = 95 - \frac{1}{2} \left[ \frac{5}{2} (x-1)^2 - (x-1)(y-10) + 2(y-10)^2 \right] \]  

(5 pts)

\[ A^2 = \frac{5}{2} (x-1)^2 \Rightarrow A = \frac{\sqrt{5}}{\sqrt{2}} (x-1) \]

\[ 2AB = 2 \frac{\sqrt{5}}{\sqrt{2}} (x-1) B = -(x-1)(y-10) \]

\[ B = -\frac{1}{\sqrt{10}} (y-10) \]

\[ p_2(x,y) = 95 - \frac{1}{2} \left[ (A + B)^2 - B^2 + 2(y-10)^2 \right] \]

\[ p_2(x,y) = 95 - \frac{1}{2} \left[ \left( \frac{\sqrt{5}}{\sqrt{2}} (x-1) - \frac{1}{\sqrt{10}} (y-10) \right)^2 + \frac{19}{10} (y-10)^2 \right] \]  

(5 pts)

\[ \cdot: \text{There is a local maximum at (1,10).} \]

**Bonus:** Find the second-order Taylor polynomial of \( f \) at the critical point and explain your answer.

\[ f(a,b) + Df(a,b) \cdot ((x,y) - (a,b)) + \frac{1}{2} \left[ f_{xx} (x-a)^2 + f_{xy} (x-a)(y-b) + f_{yy} (y-b)^2 \right] \]

\[ 95 + (0,0) \cdot ((x,y) - (1,10)) + \frac{1}{2} \left[ -\frac{10}{4} (x-1)^2 + \frac{1}{8} (x-1)(y-10) - (y-10)^2 \right] \]

\[ 95 - \frac{5}{4} (x-1)^2 + \frac{1}{8} (x-1)(y-10) - (y-10)^2 \]

The second-order Taylor polynomial of \( f \) at the critical point (1,10) uses completing the square to classify the critical point.
PROBLEM 2

Find the area of the surface $G$ cut from the hemisphere $x^2 + y^2 + z^2 = 4^2$, $z \geq 0$, by the plane $z = 1$ and $z = 3$ by setting up an integral.

Method 1 - Cylindrical Coordinates

Set up the Integral

\[
\int_{\theta=0}^{2\pi} \int_{r=\sqrt{1}}^{r=\sqrt{3}} \frac{2}{\sqrt{4^2 - r^2}} r \, dr \, d\theta
\]

(5 pts) (5 pts) (5 pts)

The surface area of the hemisphere between $z = 1$ and $z = 3$ is

\[
\iint_S \sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{\left(\frac{-x}{\sqrt{4^2 - r^2}}\right)^2 + \left(\frac{-y}{\sqrt{4^2 - r^2}}\right)^2 + 1}
\]

\[
z = \left(4^2 - x^2 - y^2\right)^{\frac{1}{2}} \quad r^2 = x^2 + y^2
\]

\[
2\pi \int_0^1 \left(\frac{2}{\sqrt{u}}\right) \left(\frac{1}{2}\right) du \quad \text{where } u = 4 - r^2, \quad du = -2rdr \quad \Rightarrow \quad \frac{1}{2}du = rdr
\]

\[
2\pi \int_1^9 u^{\frac{1}{2}} \, du
\]

\[
2\pi 2u^{\frac{3}{2}} \bigg|_1^9 = 16\pi
\]
Method 2 - Spherical Coordinates

Set up the Integral

\[ \text{SA} = \int_{\phi=0}^{\phi} \int_{\phi=\phi_0}^{\phi} 4^2 \sin \phi \, d\phi \, d\theta \]  
(5 pts)  
(5 pts)

\[ \text{SA} = 2\pi 4^2 (\cos \phi) \bigg|_{\phi_0}^{\phi} \]  
(5 pts)

\[ = 2\pi 4^2 (\cos \phi_0 - \cos \phi_1) \]

\[ \cos \phi_0 = \frac{3}{4} \text{ and } \cos \phi_1 = \frac{1}{4} \]

\[ = 2\pi 4^2 \frac{2}{4} = 16\pi \]
PROBLEM 3

Evaluate the integral by reversing the order of integration.

\[ \int_{0}^{1} \int_{x=8y}^{8} e^{x^2} \, dA \]

Set up the Integral

\[ \int_{y=0}^{1} \int_{x=8y}^{8} e^{x^2} \, dx \, dy \quad (5 \text{ pts}) \]

Reverse the order of integration

Initially slicing the graph horizontally into slices. We want to slice it with vertical lines. By examining the graph we can see that to switch we need to first integral along along the y-axis from 0 to \( \frac{1}{8} \) and then integrate along the x-axis from 0 to 8.

Now written in integral notation: \( \int_{y=0}^{8} \int_{x=0}^{\frac{1}{8}y} e^{x^2} \, dy \, dx \quad (5 \text{ pts}) \)

Evaluate the Integral

\[ \int_{0}^{8} e^{x^2} \frac{1}{8} \, x \, dx \]

\[ = \left( \frac{1}{8} \right) \left( \frac{1}{2} \right) e^{x^2} \, dx \bigg|_{0}^{8} \quad (5 \text{ pts}) \]

\[ = \frac{1}{16} \left( -1 + e^{64} \right) \]
PROBLEM 4

Using polar coordinates, set up the integral which gives the area that lies in the first quadrant between the line \( y = 1 \) and between the circle \( x^2 + y^2 = 4 \) and \( x^2 - 2x + y^2 = 0 \).

Convert to Polar Coordinates  \( \text{(5 pts)} \)

\[
\begin{align*}
x^2 + y^2 &= 4 \quad \text{a circle at the origin with a radius of 2} \quad r = 2 \\
x^2 - 2x + y^2 &= 0 \quad \text{a circle at (1,0) with a radius of 1} \quad r = 2 \cos \theta \\
y &= 1 \quad \text{horizontal line} \quad r = \frac{1}{\sin \theta}
\end{align*}
\]

Find the angle between the x-axis and where \( r = \csc \theta \) and \( r = 2 \) intersect.

\[
\frac{1}{\sin \theta_0} = \frac{1}{2} \implies \theta_0 = \sin^{-1} \left( \frac{1}{2} \right) \implies \theta_0 = \frac{\pi}{6}
\]

Set up the Integral  \( \text{(5 pts)} \)

\[
\int_{\theta=0}^{\pi/6} \int_{r=2}^{2 \cos \theta} r \, dr \, d\theta + \int_{\theta=\pi/6}^{\pi/2} \int_{r=2 \cos \theta}^{\infty} r \, dr \, d\theta
\]

\( \text{(5 pts)} \)  \( \text{(5 pts)} \)  \( \text{(5 pts)} \)
PROBLEM 5

Use spherical coordinates to evaluate the triple integral
\[
\iiint_{E} \frac{e^{-x^2 - y^2 - z^2}}{\sqrt{x^2 + y^2 + z^2}} dV,
\]
where \( E \) is the region bounded by the spheres \( x^2 + y^2 + z^2 = 25 \) and \( x^2 + y^2 + z^2 = 81 \).

Change to spherical coordinates
\[
\rho^2 = 25 \quad \rho^2 = 81 \quad (5 \text{ pts})
\]

Set up the Integral
\[
\int_{\rho=5}^{9} \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \frac{e^{-\rho^2}}{\rho} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]
(5 pts) (5 pts)

Evaluate the Integral
\[
= 2\pi \left(-\cos \phi \right)_{0}^{\pi} \int_{5}^{9} \rho e^{-\rho^2} \, d\rho
\]
\[
= 2\pi 2 \left(-\frac{1}{2} e^{-\rho^2} \right)_{5}^{9} \quad (5 \text{ pts})
\]
\[
= \frac{2 (1 + e^{56}) \pi}{e^{81}}
\]