1.10 Modeling and Graphical Analysis of Updating Functions

MATHMATICAl TECHNiques

-shirt use the idea of the weighted average to find the following.

- **EXERCISE 1.10.1**
  1.0 liters of water at 30°C is mixed with 2.0 liters of water at 100°C. What is the temperature of the resulting mixture?

- **EXERCISE 1.10.2**
  2.0 ml of water with a concentration of 0.85 moles/liter, is mixed with 5.0 ml of water with a concentration of 0.70 moles/liter. What is the concentration of the mixture?

- **EXERCISE 1.10.3**
  In a class of 52 students, 20 scored 50 on a test, 18 scored 75, and the rest scored 100. What was the average score?

- **EXERCISE 1.10.4**
  1.0 liters of water at 10°C is mixed with 2.0 liters of water at 20°C, 3.0 liters of water at 30°C, and 4.0 liters of water at 40°C. What is the temperature of the resulting mixture?

-shirt express the following weighted averages in terms of the given variables.

- **EXERCISE 1.10.5**
  1.0 liters of water at temperature $T_1$ is mixed with 2.0 liters of water at temperature $T_2$. What is the temperature of the resulting mixture? Set $T_1 = 30$ and $T_2 = 100$ and compare with the result of exercise 1.10.1.

- **EXERCISE 1.10.6**
  $V_1$ liters of water at 30°C is mixed with $V_2$ liters of water at 100°C. What is the temperature of the resulting mixture? Set $V_1 = 1.0$ and $V_2 = 2.0$ and compare with the result of exercise 1.10.1.

- **EXERCISE 1.10.7**
  $V_1$ liters of water at temperature $T_1$ is mixed with $V_2$ liters of water at temperature $T_2$. What is the temperature of the resulting mixture?

- **EXERCISE 1.10.8**
  $V_1$ liters of water at temperature $T_1$ is mixed with $V_2$ liters of water at temperature $T_2$ and $V_3$ liters of water at temperature $T_3$. What is the temperature of the resulting mixture?

-shirt the following steps are used to build a cobweb diagram. follow them for the given updating function based on bacterial populations.

  - a. Graph the updating function.
  - b. Use your graph of the updating function to find the point $(b_0, b_1)$.
  - c. Reflect it off the diagonal to find the point $(b_1, b_1)$.
  - d. Use the graph of the updating function to find $(b_1, b_2)$.
  - e. Reflect off the diagonal to find the point $(b_2, b_2)$.
  - f. Use the graph of the updating function to find $(b_2, b_3)$.
  - g. Sketch the solution as a function of time.

- **EXERCISE 1.10.9**
  The discrete-time dynamical system $b_{t+1} = 2.0b_t$ with $b_0 = 1.0 \times 10^6$.

- **EXERCISE 1.10.10**
  The discrete-time dynamical system $b_{t+1} = 0.5b_t$ with $b_0 = 1.0 \times 10^6$.

-shirt find and graph the solutions of the following discrete-time dynamical systems for 3 steps starting from the given initial condition. plot the points you find on a cobweb diagram.

- **EXERCISE 1.10.11**
  $v_{t+1} = 1.5v_t$, starting from $v_0 = 1220 \mu m^3$ (as in exercise 1.5.1).

- **EXERCISE 1.10.12**
  $l_{t+1} = l_t - 1.7$, starting from $l_0 = 13.1$ cm (as in exercise 1.5.2).
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- **EXERCISE 1.10.13**
  $n_{t+1} = 0.5n_t$, starting from $n_0 = 1200$ (as in exercise 1.5.3).
- **EXERCISE 1.10.14**
  $M_{t+1} = 0.5M_t + 1.0$ starting from the initial condition $M_0 = 18.0$ (as in exercise 1.5.4).

\[\text{♣} \text{ Graph the updating functions associated with the following discrete-time dynamical systems, and cobweb for five steps starting from the given initial condition.} \]

- **EXERCISE 1.10.15**
  $x_{t+1} = 2x_t - 1$, starting from $x_0 = 2$.
- **EXERCISE 1.10.16**
  $z_{t+1} = 0.9z_t + 1$, starting from $z_0 = 3$.
- **EXERCISE 1.10.17**
  $u_{t+1} = -0.5u_t + 3$, starting from $u_0 = 0$.
- **EXERCISE 1.10.18**
  $x_{t+1} = 4 - x_t$, starting from $x_0 = 1$ (as in exercise 1.5.15).
- **EXERCISE 1.10.19**
  $x_{t+1} = \frac{x_t}{1 + x_t}$, starting from $x_0 = 1$ (as in exercise 1.5.13).
- **EXERCISE 1.10.20**
  $x_{t+1} = \frac{x_t}{x_t - 1}$, starting from $x_0 = 3$ (as in exercise 1.5.16).

**APPLICATIONS**

\[\text{♣} \text{ Suppose that the volume of the lung is } V, \text{ the amount breathed in and out is } W \text{ and the ambient concentration is } \gamma \text{ mmol/L. For each of the given sets of parameter values and the given initial condition, find the following:} \]

  a. the amount of chemical in the lung before breathing,
  b. the amount of chemical breathed out,
  c. the amount of chemical in the lung after breathing out,
  d. the amount of chemical breathed in,
  e. the amount of chemical in the lung after breathing in,
  f. the concentration of chemical in the lung after breathing in,
  g. Compare this result with the result of using equation 1.44 (make sure to compute } q \text{ as } W/V. \]

- **EXERCISE 1.10.21**
  $V = 2.0 \text{ L}, W = 0.5 \text{ L}, \gamma = 5.0 \text{ mmol/L}, c_0 = 1.0 \text{ mmol/L}$.
- **EXERCISE 1.10.22**
  $V = 1.0 \text{ L}, W = 0.1 \text{ L}, \gamma = 8.0 \text{ mmol/L}, c_0 = 4.0 \text{ mmol/L}$.
- **EXERCISE 1.10.23**
  $V = 1.0 \text{ L}, W = 0.9 \text{ L}, \gamma = 5.0 \text{ mmol/L}, c_0 = 9.0 \text{ mmol/L}$.
- **EXERCISE 1.10.24**
  $V = 10.0 \text{ L}, W = 0.2 \text{ L}, \gamma = 1.0 \text{ mmol/L}, c_0 = 9.0 \text{ mmol/L}$.

\[\text{♣} \text{ Find and graph the updating function in the following cases. Cobweb for three steps starting from the points indicated in the earlier problems. Sketch the solutions.} \]

- **EXERCISE 1.10.25**
  The situation in exercise 1.10.21.
- **EXERCISE 1.10.26**
  The situation in exercise 1.10.22.
- **EXERCISE 1.10.27**
  The situation in exercise 1.10.23.
- **EXERCISE 1.10.28**
  The situation in exercise 1.10.24.
Cobweb the following discrete-time dynamical systems for 5 steps starting from the given initial condition.

- **EXERCISE 1.10.29**
  The tree growth system (equation 1.4) \( h_{t+1} = h_t + 1 \) with initial condition \( h_0 = 10 \). Can you see why the solution is so simple?

- **EXERCISE 1.10.30**
  The lizard-mite system (equation 1.18) \( x_{t+1} = 2x_t + 30 \) with initial condition \( x_0 = 0 \).

- **EXERCISE 1.10.31**
  In this version, a fixed amount of chemical is removed from the lung each breathe.

  a. The lung in exercise 1.10.21 absorbs \( A = 1.0 \) mmol of the chemical right before breathing out. Find the total amount and concentration right after absorption, after breathing out, and again after breathing in.

  b. Try the same steps starting from \( c_l \) rather than 1.0 mmol.

  c. Write the updating function.

  d. For what values of \( c_l \) does this derivation fail to make sense?

- **EXERCISE 1.10.32**
  Another way to incorporate absorption is for the body to absorb a particular fraction of the chemical. The following steps outline construction of this alternative model.

  a. Suppose the lung in exercise 1.10.21 absorbs 20% of the chemical in the lung just before breathing out. Find the total amount and concentration right after absorption, after breathing out, and again after breathing in.

  b. Try the same steps starting from \( c_l \) rather than 1.0 mmol.

  c. Write the updating function.

  d. Does this updating function always make sense? Why?

- **EXERCISE 1.10.33**
  A bacterial population that has per capita reproduction \( r < 1 \) but which is supplemented each generation has an updating function much like that of the lung. Use the following steps to build the updating function in the two given cases.

  a. Starting from \( 3.0 \times 10^6 \) bacteria, find the number after reproduction.

  b. Find the number after the new bacteria are added.

  c. Find the updating function.

  d. What values of \( q \) and \( \gamma \) would have produced the same updating function?

- **EXERCISE 1.10.34**
  A population of bacteria has per capita reproduction \( r = 0.6 \), and \( 1.0 \times 10^6 \) bacteria are added each generation.

- **EXERCISE 1.10.35**
  A population of bacteria has per capita reproduction \( r = 0.2 \), and \( 5.0 \times 10^6 \) bacteria are added each generation.

- **EXERCISE 1.10.36**
  Lakes receive water from streams each year, and lose water to outflowing streams and evaporation. The salt level in a lake depends greatly on the ratio of outflow to evaporation. In the following, consider a lake that receives 100 gallons of water per year with salinity of 1 part per thousand. The lake contains 10,000 gallons of water and starts with salinity 2 parts per thousand. In each case, find the following

  a. the total salt before the inflow,

  b. total water,

  c. total salt and salt concentration after inflow,

  d. total water, total salt and salt concentration after outflow or evaporation,
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e. the updating function.

- **EXERCISE 1.10.35**
  There is no evaporation, and that 100 gallons of water flow out each year.

- **EXERCISE 1.10.36**
  50 gallons flow out and 50 gallons evaporate (no salt is removed by evaporation).

- **EXERCISE 1.10.37**
  All 100 gallons evaporate, and there is no outflow.

- **EXERCISE 1.10.38**
  Assume instead that 1000 gallons flow in. All 1000 gallons evaporate and there is no outflow.
Chapter 4

Answers

1.10.1. 2/3 of the water started at 100°C, and only 1/3 started at 30°C. The final temperature is then the weighted average 
\[ T = \frac{2}{3} \cdot 100°C + \frac{1}{3} \cdot 30°C = 76.7°C. \]

1.10.3. 20 out 52 got 50, 18 out 52 got 75 and 14 out 52 got 100, for an average of 
\[ \frac{20}{52} \cdot 50 + \frac{18}{52} \cdot 75 + \frac{14}{52} \cdot 100 = 72.1. \]

1.10.5. 1/3 of the water started at \( T_1 \), and 2/3 started at \( T_2 \). The final temperature is then the weighted average 
\[ T = \frac{1}{3} \cdot T_1 + \frac{2}{3} \cdot T_2. \] If \( T_1 = 30\) and \( T_2 = 100\), we get 
\[ T = \frac{1}{3} \cdot 30 + \frac{2}{3} \cdot 100 = 76.7 \] as before.

1.10.7. A fraction \( V_1 / (V_1 + V_2) \) started at \( T_1 \), and a fraction \( V_2 / (V_1 + V_2) \) started at \( T_2 \). The final temperature is the weighted average 
\[ T = T_1 \frac{V_1}{V_1 + V_2} + T_2 \frac{V_2}{V_1 + V_2}. \]

1.10.9.

1.10.11. The solution was \( V_t = 1.5^t \cdot 1220 \text{m}^3 \).

1.10.13. \( n_t = 0.5^t \cdot 1200. \)
1.10.15.

1.10.17.

1.10.19.

1.10.21.

a. amount = volume times concentration $Vc_0 = 2.0 \text{ mmol}$.

b. 0.5 L at 1.0 mmol/L = 0.5 mmol.

c. 1.5 L at 1.0 mmol/L = 1.5 mmol.

d. 0.5 L at 5.0 mmol/L = 2.5 mmol.

e. 1.5 + 2.5 = 4.0 mmol

f. 4.0 mmol/2.0 L = 2.0 mmol/L

g. $q = 0.5/2.0 = 0.25$. Then $c_{t+1} = (1-q)c_t + q\gamma = 0.75\cdot c_t + 0.25\cdot 5.0$. When $c_0 = 1.0, c_1 = 0.75\cdot 1.0 + 0.25\cdot 5.0 = 2.0 \text{ mmol/L}$.  

1.10.23. Start with 9.0 mmol, breathe out 8.1 mmol, leaving 0.9 mmol, breathe in 4.5 mmol, ending with 5.4 mmol, and a concentration of 5.4 mmol/L. This checks with the the updating function approach. In this case, \( q = 0.9 \) and \( \gamma = 5.0 \), so \( c_{t+1} = 0.1c_t + 0.9 \cdot 5.0 \). Substituting \( c_0 = 9.0 \), we find \( c_1 = 5.4 \).

1.10.25. The updating function has \( q = 0.5/2.0 = 0.25 \) and \( \gamma = 5.0 \), and thus has formula \( c_{t+1} = (1 - 0.25)c_t + 0.25 \cdot 5.0 = 0.75c_1 + 1.25 \). We want to start from 1.0 mmol/L.

1.10.27. The updating function is \( c_{t+1} = 0.1c_t + 4.5 \), starting from \( c_0 = 9.0 \).

1.10.29. The solution is simple because the graph of the updating function is parallel to the diagonal.

1.10.31.

a. Starts with 2.0 mmol, then 1.0 mmol is absorbed leaving 1.0 mmol and a concentration of 0.5 mmol/L. 0.5 L of air with this concentration are breathed out, taking with them 0.25 mmol, leaving 0.75 mmol. 0.5 L are breathed in with concentration 5.0, adding 2.5 to the remaining 0.75 for a total of 3.25 mmol, and a concentration of 1.625 mmol/L.

b. Starts with 2.0\(c_t\), has 2.0\(c_t - 1.0\) after absorption, and a concentration of \(c_t - 0.5\). Exhaled air then has \(0.5c_t - 0.25\) mmol, leaving \(1.5c_t - 0.75\) in the lung. Inhaled air always carries 2.5 mmol in, for a total of \(1.5c_t + 1.75\) and a concentration of \(0.75c_t + 0.875\).

c. \(c_{t+1} = 0.75c_t + 0.875\).

d. The value \(2.0c_t - 1.0\) is negative if \(c_t < 0.5\). The lung cannot absorb 1.0 mmol if there is not 1.0 mmol to absorb.

1.10.33.

a. Population after reproduction is \(0.6 \cdot 3.0 \times 10^6 = 1.8 \times 10^6\).

b. Population after supplementation is \(1.8 \times 10^6 + 1.0 \times 10^6 = 2.8 \times 10^6\).
c. \( b_{t+1} = 0.6b_t + 1.0 \times 10^6 \).

d. It should look like \( b_{t+1} = (1 - q)b_t + q\gamma \). To match the first term, we need \( 1 - q = 0.6 \) or \( q = 0.4 \). To match the second term, we need \( q\gamma = 1.0 \times 10^6 \) or \( 0.4\gamma = 1.0 \times 10^6 \) which has solution \( \gamma = 2.5 \times 10^6 \).

1.10.35. Start with 20 gallons of salt, get 0.1 gallons of salt, so there are 20.1 gallons of salt in 10,100 gallons of water, or 1.99 parts per thousand. Then 0.199 gallons of salt leave in 100 gallons, leaving a total of 19.9 gallons of salt and a concentration of 1.99 parts per thousand. Starting from a salt concentration of \( s_t \) in parts per thousand, have 100 gallons of salt. After inflow, there are \( 100s_t + 0.1 \) gallons in 10,100 gallons of water, for a concentration of \( 0.99s_t + 0.0099 \). Then \( 0.99s_t + 0.0099 \) gallons of salt flow out, leaving \( 9.9s_t + 0.099 \) gallons of salt and a concentration of \( 0.99s_t + 0.0099 \). So \( s_{t+1} = 0.99s_t + 0.0099 \).

1.10.37. Start with 20 gallons of salt, get 0.1 gallons of salt, so there are 20.1 gallons of salt in 10,100 gallons of water, or 1.99 parts per thousand. Then no salt leaves by evaporation, so there are 20.1 gallons in 10,000 gallons or a concentration of 2.01 parts per thousand. The updating function will be \( s_{t+1} = s_t + 0.01 \).