Fourier Transform for Partial Differential Equations

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Introduction: Fourier Transform

The Fourier transform creates another representation of a signal, specifically a representation as a weighted sum of complex exponentials. It is designed for non-periodic signals that decay at infinity, the condition that $\int_{-\infty}^{\infty} |f(x)| \, dx$ is finite.

Because of Euler’s formula

$$e^{iq} = \cos(q) + i \sin(q)$$

where $i^2 = -1$, the Fourier transform produces a representation of a signal (or an image) as a weighted sum of sines and cosines.

Given a signal (or image) $a$ and its Fourier transform $A$, then the forward Fourier transform goes from the spatial domain, either continuous or discrete, to the frequency domain, which is always continuous. The inverse Fourier transform goes from the frequency domain back to the spatial domain.

Forward: $A = F(a)$, Inverse: $a = F^{-1}(A)$
Definition: Fourier Transform

\[ F(w) = FT[f](w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{iwx} \, dx \]

\[ f(x) = FT[f]^{-1}(x) = \int_{-\infty}^{\infty} F(w) e^{-iwx} \, dw \]

The **Reciprocity Relation** connects the two similar formulas:

\[ f(-u) = \int_{-\infty}^{\infty} F(w) e^{iwu} \, dw \]

\[ = 2\pi \frac{1}{2\pi} \int_{-\infty}^{\infty} F(x) e^{iux} \, dx \]

\[ = 2\pi FT[F](u) \]

In words, setting \( x = -u \) in the inverse Fourier transform equation produces the forward Fourier transform equation for the function \( F \) multiplied by \( 2\pi \).
Fourier Transform Properties

1. Linearity \( FT[f + g] = FT[f] + FT[g] \) and \( FT[cf] = cFT[f] \)

2. \( x \)-differentiation \( FT[f'] = (-iw)FT[f] \)

3. \( w \)-differentiation \( FT[xf(x)] = -i \frac{d}{dw}FT[f] \)

4. Convolution \( FT[f * g] = FT[f]FT[g] \), where
\[
f * g(x) = g * f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x - t)g(t)dt
\]

5. \( x \)-shifting \( FT[f(x - a)] = e^{iaw}FT[f] \)

6. \( w \)-shifting \( FT[e^{iax}f(x)](w) = FT[f](w + a) \)
Parseval’s Energy Identity

The square $|f(t)|^2$ of the time signal represents how the energy contained in the signal distributes over time $t$, while the spectrum squared $|F(w)|^2$ represents how the energy distributes over frequency (the power density spectrum). The same amount of energy is contained in either time or frequency domain, because of Parseval’s formula:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |F(w)|^2 dw$$

To evaluate what this means graphically, compute both integrands and graph them on a large interval. Use the Gaussian example

$$f(t) = e^{-\alpha t^2}$$
Fourier Sine and Cosine Integral Representations

The theory applies to functions $f(x)$ defined only on $0 < x < \infty$.

**Definition.** The Fourier Cosine Integral Representation

$$f(x) = \int_0^\infty A(w) \cos(wx) \, dw, \quad A(w) = \frac{2}{\pi} \int_0^\infty f(t) \cos(wt) \, dt$$

**Definition.** The Fourier Sine Integral Representation

$$f(x) = \int_0^\infty B(w) \sin(wx) \, dw, \quad B(w) = \frac{2}{\pi} \int_0^\infty f(t) \sin(wt) \, dt$$
Fourier Sine and Cosine Transforms

Definition. The Fourier Cosine and Sine Forward Transforms

\[ FCT[f](w) = \frac{2}{\pi} \int_0^{\infty} f(t) \cos(\omega t) \, dt, \]
\[ FST[f](w) = \frac{2}{\pi} \int_0^{\infty} f(t) \sin(\omega t) \, dt \]

Definition. The Fourier Cosine and Sine Inverse Transforms

\[ f(x) = \int_0^{\infty} FCT[f](w) \cos(\omega x) \, dw, \]
\[ f(x) = \int_0^{\infty} FST[f](w) \sin(\omega x) \, dw \]
Fourier Sine and Cosine Transform Properties

Theorem 1 (Properties)

- \( FCT[f](w) = 2FT[f](w) \) for \( w \geq 0 \), provided \( f \) is even on \((-\infty, \infty)\).
- \( FST[f](w) = 2FT[f](w) \) for \( w \geq 0 \), provided \( f \) is odd on \((-\infty, \infty)\).
- Both \( FCT \) and \( FST \) satisfy the Fourier transform’s linearity property.
- \( FCT[f'] = w FST[f] - \frac{2}{\pi} f(0) \), provided \( \lim_{x \to \infty} f(x) = 0 \).
- \( FST[f'] = -w FCT[f] \), provided \( \lim_{x \to \infty} f(x) = 0 \).
- \( FCT[xf(x)] = \frac{d}{dw} FST[f] \)
- \( FST[xf(x)] = -\frac{d}{dw} FCT[f] \)