Chapter H10: 10.4, 10.5 Fourier Transform

EXERCISES H10.4, Fourier Transform and the Heat Equation

Problem H10.4-2. (Heat Equation on $-\infty < x < \infty$, Limit Zero at Infinity)
For the heat equation,
\[ u(x, t) = \int_{-\infty}^{\infty} F(w)e^{-iwx}e^{-kw^2t}dw. \]
Show that $\lim_{x \to \infty} u(x, t) = 0$ even though $\phi(x) = e^{-iwx}$ does not decay as $x \to \infty$. (Hint: Integrate by parts.)

Problem H*10.4-3. (Diffusion-Convection Equation)
(a) Solve the diffusion equation with convection:
\[ u_t(x, t) = ku_{xx}(x, t) + cu_x(x, t), \quad -\infty < x < \infty, \quad t > 0, \]
subject to $u(x, 0) = f(x)$.
[Hint: Use the convolution theorem and the shift theorem (see Exercise H10.4-5).]
(b) Consider the initial condition to be the Dirac unit impulse $\delta(t)$. Sketch the corresponding diffusion-convection solution $u(x, t)$ for various values of $t > 0$. Comment on the significance of the convection term $cu_x(x, t)$.

Problem H*10.4-5. (Diffusion Equation with Source $Q(x, t)$)
Consider the diffusion equation
\[ u_t(x, t) = ku_{xx}(x, t) + Q(x, t), \quad -\infty < x < \infty, \quad t > 0, \]
with initial condition $u(x, 0) = f(x)$.
(a) Show that a particular solution for the Fourier transform $U(w) = \mathcal{F}[u(x, t)]$ is
\[ U_1(w) = e^{-kw^2t} \int_0^t Q_1(w, r)e^{kw^2r}dr, \quad Q_1(w, t) = \mathcal{F}[Q(x, t)]. \]
(b) Determine $U_1$.
*(c) Solve for $u(x, t)$ (in the simplest form possible).
\textbf{Answer}: $u(x, t)$ is the Heat kernel solution $u_0(x, t)$ of the homogeneous problem plus the inverse Fourier transform of $U_1$, which is
\[ u_1(x, t) = \frac{1}{2\pi} \int_0^t \int_{-\infty}^{\infty} Q(v, t) \sqrt{\frac{\pi}{k(t-r)}} e^{-\frac{(x-v)^2}{4k(t-r)}}dvdr. \]

Problem XC-H10.4-11. (Fourier Transform of a Product)
Derive an expression for the Fourier transform of the product $f(x)g(x)$.
\textbf{Answer}: $\mathcal{F}[f(x)g(x)]$ is the convolution of $\mathcal{F}[f(x)]$ with $\mathcal{F}[g(x)]$. 