Chapter H10: 10.2, 10.3, 10.4 Fourier Transform

EXERCISES H10.2

Problem H*10.2-1. (Heat Equation on \(-\infty < x < \infty\), Coefficient Identities)
Determine complex \(c(w)\) so that
\[
u(x,t) = \int_{-\infty}^{\infty} c(w) e^{-iwx} e^{-kw^2t} dw
\]
is equivalent to
\[
u(x,t) = \int_{0}^{\infty} \left( A(w) \cos(wx) + B(w) \sin(wx) \right) e^{-kw^2t} dw
\]
with real \(A(w)\) and \(B(w)\). Then show that \(c(-w) = \bar{c}(w)\), \(w > 0\), where the over-bar denotes the complex conjugate.

Problem H10.2-2. (Heat Equation, Complex Integrand)
If \(c(-w) = \bar{c}(w)\) (see the preceding exercise), then show that \(u(x,t)\) is real, where
\[
u(x,t) = \int_{-\infty}^{\infty} c(w) e^{-iwx} e^{-kw^2t} dw
\]

EXERCISES H10.3

Problem H10.3-1. (Linearity of the Fourier Transform)
Show that the Fourier transform is a linear operator; that is, show that
(a) \(FT[c_1 f(x) + c_2 g(x)] = c_1 FT[f(x)] + c_2 FT[g(x)]\)
(b) \(FT[f(x)g(x)] \neq FT[f(x)] FT[g(x)]\)

Problem H10.3-2. (Linearity of the Inverse Fourier Transform)
Show that the inverse Fourier transform is a linear operator; that is, show that
(a) \(FT^{-1}[c_1 FT[f(x)] + c_2 FT[g(x)]] = c_1 f(x) + c_2 g(x)\)
(b) \(FT^{-1}[F(w)G(w)] \neq f(x)g(x)\)

Problem H10.3-3. (Complex Conjugate and Fourier Transform)
Let \(F(w)\) be the Fourier transform of \(f(x)\). Show that if \(f(x)\) is real, then \(F^*(w) = F(-w)\), where * denotes the complex conjugate.

Problem XC-H10.3-4. (Transforms of Functions Depending on a Parameter \(\alpha\))
Show that \(FT\left[\int f(x; \alpha) d\alpha\right] = \int F(w, \alpha) d\alpha\).

Problem H10.3-5. (Shift and the Fourier Transform)
If \(F(w)\) is the Fourier transform of \(f(x)\), show that the inverse Fourier transform of \(e^{iwx} F(w)\) is \(f(x - \beta)\). This result is known as the shift theorem for Fourier transforms.

Problem H*10.3-6. (Transform of the Unit Pulse: Sinc Function, Rect Pulse)
If \( f(x) = \begin{cases} 0 & |x| > a, \\ 1 & |x| < a, \end{cases} \) then determine the Fourier transform of \( f(x) \).

**Answer:** \( \frac{1}{\pi} \sin(aw) \), or \( \frac{a}{\pi} \text{sinc}(aw) \). The \text{sinc} function is a widely researched function in numerical analysis, defined by \( \text{sinc}(u) = \frac{\sin(u)}{u} \).

**Remark.** A standard transform table may contain instead the function \text{rect} , a rectangular pulse of width 1 with value \( \frac{1}{2} \) at \( x = \pm \frac{1}{2} \).

[The answer is given in the table of Fourier transforms in H10, Section 4.4.]

**Problem H*10.3-7. (Transform Table, Exponential Transform)**
If \( F(w) = e^{-|w|^\alpha}, \alpha > 0 \), then determine the inverse Fourier transform of \( F(w) \).

**Answer:** \( f(x) = \text{FT}^{-1}(F(w)) = \frac{2\alpha}{x^2 + \alpha^2} \).

[The answer is given in the table of Fourier transforms in H10, Section 4.4.]

**Problem XC-H10.3-8. (Multiply by \( x \) and Differentiation of \( F(w) \))**
If \( F(w) \) is the Fourier transform of \( f(x) \), show that \(-idF/dw\) is the Fourier transform of \( xf(x) \).

**Problem XC-H10.3-9. (Textbook Details)**
(a) Multiply (10.3.6) (assuming that \( \gamma = 1 \)) by \( e^{-iwx} \) and integrate from \(-L\) to \(L\) to show that

\[
\int_{-L}^{L} F(w)e^{-iwx}\,dw = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \frac{2\sin(L(u-x))}{u-x} \,du. \tag{10.3.13}
\]

(b) Derive (10.3.7). For simplicity, assume that \( f(x) \) is continuous. [Hints: Let \( f(u) = f(x) + f(u) - f(x) \). Use the sine integral, \( \int_{0}^{\infty} \frac{\sin(s)}{s} \,ds = \frac{\pi}{2} \). Integrate (10.3.13) by parts and then take the limit as \( L \to \infty \).]

**Problem XC-H10.3-11. (Scaling)**
(a) If \( f(x) \) is a function with unit area, \( \int_{-\infty}^{\infty} f(x)\,dx = 1 \), show that the scaled and stretched function \( (1/\alpha)f(x/\alpha) \) also has unit area.

(b) If \( F(w) \) is the Fourier transform of \( f(x) \), show that \( F(\alpha w) \) is the Fourier transform of \( (\alpha)f(x) \).

(c) Show that part (b) implies that broadly spread functions have sharply peaked Fourier transforms near \( w = 0 \), and vice versa.

**Problem XC-H10.3-13. (Cosine)**
Evaluate \( \int_{0}^{\infty} e^{-kw^2} \cos(wx)\,dw \) in the following way. Determine \( \partial I/\partial x \), and then integrate by parts.

**Problem H10.3-14. (Gamma Function)**
The gamma function \( r(x) \) is defined as follows:

\[
\Gamma(x) = \int_{0}^{\infty} t^{x-1}e^{-t}\,dt.
\]

Show that

(a) \( \Gamma(1) = 1 \) \hspace{1cm} (b) \( \Gamma(x+1) = \Gamma(x) \)

(c) \( \Gamma(n+1) = n! \) \hspace{1cm} (d) \( \Gamma(1/2) = 2\int_{0}^{\infty} e^{-t^2}\,dt = \sqrt{\pi} \)

(e) What is \( \Gamma(3/2) \)?
(a) Using the definition of the gamma function in the previous Exercise, show that
\[ \Gamma(x) = 2 \int_0^\infty u^{2x-1} e^{-u^2} du. \]

(b) Using double integrals in polar coordinates, show that
\[ \Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}. \]

[Hint: It is known from complex variables that \( 2 \int_0^{\pi/2} (\tan \theta)^{2x-1} d\theta = \frac{\pi}{\sin(\pi z)} \).]

Problem XC-H*10.3-16. (Gamma Function Identity)
Evaluate \( \int_0^\infty y^p e^{-ky^n} dy \) in terms of the gamma function (see Exercise 10.3.14).