Math 3150 Problems  
Haberman Chapter H4

Due Date: Problems are collected on Wednesday.

Chapter H4: 4.2, 4.3, 4.4 Vibrating String, Boundary Conditions, Fixed Ends BVP

EXERCISES H4.2

Problem H4.2-1. (One-Dimensional String Derivation)
(a) Using the equation
\[ \rho_0(x) \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + Q(x,t)\rho_0(x), \]
compute the sagged equilibrium position \( u_E(x) \) if \( Q(x,t) = -g \). Use boundary conditions \( u(0) = 0 \) and \( u(L) = 0 \) with the equilibrium equation \( 0 = T_0 \frac{\partial^2 u}{\partial x^2} - g\rho_0 \) (formally, \( u_t \equiv 0 \)).
(b) Show that \( v(x,t) = u(x,t) - u_E(x) \) satisfies
\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c^2 = \frac{T_0}{\rho_0(x)}. \]
Use superposition and the equation \( 0 = u_{xx} - gc^2 \), valid for \( u = U_E \) according to part (a).

Problem H4.2-2. (Wave Speed \( c \))
Show that \( c^2 = \frac{T_0}{\rho_0} \) has the dimensions of velocity squared.

Problem XC-H4.2-3. (String Derivation)
Consider a particle whose x-coordinate (in horizontal equilibrium) is designated by \( \alpha \). If its vertical and horizontal displacements are \( u \) and \( v \), respectively, determine its position \( x \) and \( y \). Then show that
\[ \frac{dy}{dx} = \frac{\partial u/\partial \alpha}{1 + \partial v/\partial \alpha}. \]

Problem XC-H4.2-4. (String Derivation)
Derive equations for horizontal and vertical displacements without ignoring \( v \). Assume that the string is perfectly flexible and that the tension is determined by an experimental law.

Problem XC-H4.2-5. (String Derivation, Constant Wave Speed)
Derive the partial differential equation for a vibrating string in the simplest possible manner. You may assume the string has constant mass density \( \rho_0 \), you may assume the tension \( T_0 \) is constant, and you may assume small displacements (with small slopes).
**EXERCISES H4.3**

**Problem XC-H4.3-1. (Boundary Conditions)**

If \( m = 0 \) in the model

\[
m \frac{d^2 u}{dt^2} (0, t) = -k(u(0, t) - y_s(t) - L) + T_0 \frac{\partial u}{\partial x} (0, t) + g(t)
\]

then which of the diagrams for the right end of the string shown in the figure can possibly be correct? Briefly explain. Assume that the mass can move only vertically.

![Diagram](image)

(a) ![Diagram](image) (b)

**EXERCISES H4.4**

**Problem H4.4-1. (One-Dimensional String, Constant Wave Speed \( c \))**

Consider vibrating strings of uniform density \( \rho_0 \) and tension \( T_0 \). Submit only (a) and (b).

*(a) What are the natural frequencies of a vibrating string of length \( L \) fixed at both ends?*

*(b) What are the natural frequencies of a vibrating string of length \( H \), which is fixed at \( x = 0 \) and "free" at the other end [i.e., \( \partial u/\partial x(H, t) = 0 \)]? Sketch a few modes of vibration as in Fig. 1, H4.4.

(c) Show that the modes of vibration for the odd harmonics (i.e., \( n = 1, 3, 5, \ldots \)) of part (a) are identical to modes of part (b) if \( H = L/2 \). Verify that their natural frequencies are the same. Briefly explain using symmetry arguments.

**Problem H4.4-2. (Time-Dependent String Equation)**

In Sec. H4.2 it was shown that the displacement \( u \) of a nonuniform string satisfies

\[
\rho_0(x) \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + Q(x, t)\rho_0(x),
\]

where \( Q \) represents the vertical component of the body force per unit length. If \( Q = 0 \), the partial differential equation is homogeneous. A slightly different homogeneous equation occurs if \( Q = \alpha u \).

Submit only part (c).

(a) Show that if \( \alpha < 0 \), the body force is restoring (toward \( u = 0 \)). Show that if \( \alpha > 0 \), the body force tends to push the string further away from its unperturbed position \( u = 0 \).

(b) Separate variables if \( \rho_0(x) \) and \( \alpha(x) \) are assumed to depend on \( x \), but \( T_0 \) is constant for physical reasons. Analyze the time-dependent ordinary differential equation.

*(c) Specialize part (b) to the constant coefficient case. Solve the initial value problem if \( \alpha < 0 \):

\[
\begin{align*}
    u(0, t) &= 0, & u(L, t) &= 0, \\
    u(x, 0) &= 0, & \frac{\partial u}{\partial t} (x, 0) &= f(x).
\end{align*}
\]

What are the frequencies of vibration?
Problem XC-H4.4-3. (Damped Vibrations of a One-Dimensional String)
Consider a slightly damped vibrating string that satisfies
\[ \rho(x) \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}. \]

(a) Briefly explain why \( \beta > 0. \)
*(b) Determine the solution (by separation of variables) that satisfies the boundary conditions
\[ u(0, t) = 0 \]
and the initial conditions
\[ u(x, 0) = f(x) \] and \( \frac{\partial u}{\partial t}(x, 0) = g(x). \)
You can assume that this frictional coefficient \( Q \) is relatively small \( (\beta^2 < 4\pi^2 \rho_0 T_0 / L^2). \)

Problem XC-H4.4-4. (Eigenfunction Expansion Method)
Redo Exercise 3(b), H4.4, by the eigenfunction expansion method.

Problem XC-H4.4-5. (Damped Vibrations)
Redo Exercise 3(b), H4.4, if \( 4\pi^2 \rho_0 T_0 / L^2 < \beta^2 < 16\pi^2 \rho_0 T_0 / L^2. \)

Problem XC-H4.4-6. (d’Alembert Solution)
For the classical string vibration problem with clamped ends, use the series solution for \( u(x, t) \) to show that
\[ u(x, t) = R(x - ct) + S(x + ct), \]
where \( R \) and \( S \) are some functions.

Problem H4.4-7. (d’Alembert Solution)
If a vibrating string satisfying the one-dimensional string equation with fixed ends is initially at rest, \( g(x) = 0, \) with shape \( f(x) \) given, then show that
\[ u(x, t) = \frac{1}{2} [F(x - ct) + F(x + ct)], \]
where \( F(x) \) is the odd periodic extension of \( f(x). \)
Hints.
1. For all \( x, F(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/L). \)
2. \( \sin a \cos b = \frac{1}{2} [\sin(a + b) + \sin(a - b)]. \)
Comment: This result shows that the practical difficulty of summing an infinite number of terms of a Fourier series may be avoided for the one-dimensional wave equation.

Problem XC-H4.4-8. (d’Alembert Solution)
If a vibrating string satisfying the one-dimensional string equation with fixed ends is initially unperturbed, \( f(x) = 0, \) with the initial velocity \( g(x) \) given, then show that
\[ u(x, t) = \frac{1}{2c} \int_{x - ct}^{x + ct} G(u)du, \]
where \( G(x) \) is the odd periodic extension of \( g(x). \)
Hints:
1. For all \( x, G(x) = \sum_{n=1}^{\infty} B_n \frac{\cos(n\pi x/L)}{\sin(n\pi x/L)}. \)
2. \( \sin a \sin b = \frac{1}{2} [\cos(a - b) - \cos(a + b)]. \)
See the comment after Exercise 7, H4.4.

Problem XC-H4.4-9. (Energy Conservation)
From \( u_{tt} = c^2 u_{xx}, \) derive conservation of energy for a vibrating string,
\[ \frac{dE}{dt} = c^2 u_x(x, t)u_t(x, t)|_{x=L}, \]
where the total energy \( E \) is the sum of the kinetic energy, defined by \( \int_0^L \frac{1}{2}(u_t)^2 dx, \) and the potential energy, defined by \( \int_0^L \frac{c^2}{2}(u_x)^2 dx. \)
Problem XC-H4.4-10. (Total Energy)
What happens to the total energy $E$ of a vibrating string (see Exercise 9, H4.4)
(a) If $u(0,t) = 0$ and $u(L,t) = 0$
(b) If $u_x(0,t) = 0$ and $u(L,t) = 0$
(c) If $u(0,t) = 0$ and $u_x(L,t) = -\gamma u(L,t)$ with $\gamma > 0$
(d) If $\gamma < 0$ in part (c)

Problem H4.4-11. (Potential and Kinetic Energies)
Show that the potential and kinetic energies (defined in Exercise 9, H4.4) are equal for a traveling wave, $u = R(x - ct)$.

Problem XC-H4.4-12. (Uniqueness)
Using
$$\frac{dE}{dt} = c^2 u_x(x,t)u_t(x,t)|_{x=0}^L,$$
prove that the solution of the one-dimensional string equation with fixed ends is unique.
Remark. The result means that the solution can be computed with numerical software.

Problem XC-H4.4-13. (Modes and Energies)
(a) Using
$$\frac{dE}{dt} = c^2 u_x(x,t)u_t(x,t)|_{x=0}^L,$$
calculate the energy of one normal mode.
(b) Show that the total energy, when $u(x,t)$ is a superposition of product solutions representing the solution, is the sum of the energies contained in each mode.