Chapter H3: 3.4 Differentiation and Integration of Fourier Series

Problem H3.4.1. (Integration by Parts)
The integration-by-parts formula
\[ \int_a^b u \, dv = uv \bigg|_a^b - \int_a^b v \, du \]
is known to be valid for functions \( u(x) \) and \( v(x) \) which are continuous and have continuous first derivatives. However, we will assume that \( u, v, du/dx, \) and \( dv/dx \) are continuous only for \( a < x < c \) and \( c < x < b; \) we assume that all quantities may have a jump discontinuity at \( x = c. \)

*(a) Derive an expression for \( \int_a^b u \, dv \) in terms of \( \int_a^b v \, du. \)
(b) Show that this reduces to the integration-by-parts formula if \( u \) and \( v \) are continuous across \( x = c. \) It is not necessary for \( du/dx \) and \( dv/dx \) to be continuous at \( x = c. \)

Problem H3.4.2. (Fourier Series Differentiation)
Suppose that \( f(x) \) and \( df/dx \) are piecewise smooth. Prove that the Fourier series of \( f(x) \) can be differentiated term by term if the Fourier series of \( f(x) \) is continuous.

Problem H3.4.3. (Fourier Coefficients of \( f' \))
Suppose that \( f(x) \) is continuous [except for a jump discontinuity at \( x = x_0; f(x_0-) = \alpha \) and \( f(x_0+) = \beta \)] and \( df/dx \) is piecewise smooth.

*(a) Determine the Fourier sine series of \( df/dx \) in terms of the Fourier cosine series coefficients of \( f(x). \)
(b) Determine the Fourier cosine series of \( df/dx \) in terms of the Fourier sine series coefficients of \( f(x). \)

Problem XC-H3.4-4. (Differentiation of Fourier Sine and Cosine Series)
Suppose that \( f(x) \) and \( df/dx \) are piecewise smooth.

(a) Prove that the Fourier sine series of a continuous function \( f(x) \) can only be differentiated term by term if \( f(0) = 0 \) and \( f(L) = 0. \)
(b) Prove that the Fourier cosine series of a continuous function \( f(x) \) can be differentiated term by term.

Problem XC-H3.4-5. (Fourier Cosine Series by Differentiation of a Sine Series)
Determine the Fourier cosine series of \( \sin(\pi x/L), \) using Fourier sine series formulas developed in the previous section.

Problem H3.4.6. (Series Differentiation Mistakes)
There are some things wrong in the following demonstration. Find the mistakes and correct them.

In this exercise we attempt to obtain the Fourier cosine coefficients of \( e^x: \)
\[ e^x = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x/L). \]

Differentiating yields
\[ e^x = -\sum_{n=1}^{\infty} A_n \frac{n\pi}{L} \sin(n\pi x/L), \]
the Fourier sine series of $e^x$. Differentiating again yields

$$e^x = -\sum_{n=1}^{\infty} \left( \frac{n\pi}{L} \right)^2 A_n \cos(n\pi x/L).$$

Next, we match coefficients in the first and third series, because they are both cosine series. Thus

$$A_0 = 0, \quad A_n = 0,$$

When correcting the mistakes, you should be able to obtain $A_0$ and $A_n$ without using the typical technique, that is, $A_n = (2/L) \int_0^L e^x \cos(n\pi x/L) dx$.

**Problem XC-H3.4-7. (Fourier Series with Extra Time Variable)**

Prove that the Fourier series of a continuous function $u(x, t)$ can be differentiated term by term with respect to the parameter $t$ if $\partial u/\partial t$ is piecewise smooth.

**Problem H3.4-8. (Insulated Rod Superposition Series Justification)**

Consider

$$u_t = ku_{xx}$$

subject to $u_x = 0$ at $x = 0$ and $x = L$, and $u(x, 0) = f(x)$. Solve in the following way. Look for the solution as a Fourier cosine series. Assume that $u$ and $u_x$ are continuous and $u_{xx}$ and $u_t$ are piecewise smooth. Justify all differentiations of infinite series.

**Problem H3.4-9. (Ice-Pack Rod with Heat Source $q(x, t)$)**

Consider the heat equation with a known source $q(x, t)$:

$$u_t = ku_{xx} + q(x, t)$$

with $u(0, t) = 0$ and $u(L, t) = 0$. Assume that $q(x, t)$ (for each $t > 0$) is a piecewise smooth function of $x$. Also assume that $u$ and $u_x$ are continuous functions of $x$ (for $t > 0$) and $u_{xx}$ and $u_t$ are piecewise smooth. Thus,

$$u(x, t) = \sum_{n=1}^{\infty} b_n(t) \sin(n\pi x/L).$$

What ordinary differential equation does $b_n(t)$ satisfy? Do not solve this differential equation.

**Problem XC-H3.4-10. (Insulated Rod with Heat Source $q(x, t)$)**

Modify the previous exercise if instead $u_x = 0$ at $x = 0$ and $x = L$.

**Problem XC-H3.4-11. (Ice-Pack Rod with Steady Heat Source $g(x)$)**

Consider the nonhomogeneous heat equation (with a steady heat source):

$$u_t = ku_{xx} + g(x).$$

Solve this equation with the initial condition $u(x, 0) = f(x)$ and the boundary conditions $u(O, t) = 0$ and $u(L, t) = 0$. Assume that a continuous solution exists (with continuous derivatives). [Hints: Expand the solution as a Fourier sine series (i.e., use the method of eigenfunction expansion). Expand $g(x)$ as a Fourier sine series. Solve for the Fourier sine series of the solution. Justify all differentiations with respect to $x$.]

**Problem XC-H3.4-12. (Insulated Rod with Explicit Heat Source $q(x, t)$)**

Solve the following nonhomogeneous problem:

$$u_t = ku_{xx} + e^{-t} + e^{-2t} \cos(3\pi x/L) \quad [\text{assume that } 2 \neq k(3\pi/L)^2]$$

subject to $u_x = 0$ at $x = 0$ and $x = L$, and $u(x, 0) = f(x)$. Use the following method. Look for the solution as a Fourier cosine series. Justify all differentiations of infinite series (assume appropriate continuity).

**Problem XC-H3.4-13. (Rod with Specified Endpoint Temperatures)**

Consider

$$u_t = ku_{xx}$$

subject to $u(0, t) = A(t)$, $u(L, t) = 0$, and $u(x, 0) = g(x)$. Assume that $u(x, t)$ has a Fourier sine series. Determine a differential equation for the Fourier coefficients (assume appropriate continuity).