Differential Equations 5420
Midterm Exam 1, Spring 2003
Due Date: January 31, 2003

Instructions. The four problems below are take-home, due on the date above. Answer checks are expected. If maple assist is used, then please attach the maple output.

1. (Matrix Exponential) Prove that the matrix series $\sum_{n=0}^{\infty} A^n/n!$ converges. Do it by showing that each element of the partial sum matrix $S_N = I + \cdots + A^N/N!$ corresponds to a Cauchy sequence of real numbers.

2. (Exponential identities) Prove that: (1) $AB = BA$ implies $e^{A+B} = e^A e^B = e^B e^A$, (2) $A = \text{diag}(A_1, A_2, A_3)$ implies $e^A = \text{diag}(e^{A_1}, e^{A_2}, e^{A_3})$, (3) $e^N = I + \cdots + N^k/k!$ provided $N^{k+1} = 0$.

3. (Eigenanalysis) Find the eigenvalues, eigenvectors and generalized eigenvector chains for the matrix

$$A = \begin{bmatrix}
12 & 25 & 0 & 0 & 0 & 0 & 0 \\
-5 & -8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}$$

Then solve $X' = AX$.

4. (Solving $X' = AX$) Give an example of a system $X' = AX$ with $A$ real $4 \times 4$ having solution components involving only $\sin 2t$, $\cos 2t$, $t \sin 2t$, $t \cos 2t$. Please display $A$, solve the system $X' = AX$ and verify the solution.

5. (Finite dimensional spectral theory) Using Hirsch-Smale as a reference, write a summary of finite dimensional spectral theory, as it applies to the problem $X' = AX$ with $A$ $n \times n$ real. Describe in detail how to form the real matrix $P$ of generalized eigenvectors from the chains of generalized eigenpairs. Include one illustration which computes from $A$ the chains, $P$ and the real Jordan form, followed by solving $X' = AX$. 