Name: ___________________________ U. ID: ____________

Instructions: This is a closed book, but open written notes test. Calculators are not allowed. You need to show all the details of your work to receive full credits.

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1. Verify that the function

\[ u(x, t) = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}} \]

is a solution of the heat equation

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0. \]

What will happen to \( u \) when \( x \to \pm \infty \)? Show the tendency of \( u \) as functions of \( x \) when \( t \to 0+ \), by sketching snapshots of \( u \) for several diminishing values of \( t \).

\[
\frac{\partial u}{\partial t} = \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \left( \frac{x^2}{4t^2} \right) - \frac{1}{2t^{3/2}} e^{-\frac{x^2}{4t}} = \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \left[ \frac{x^2}{4t^2} - \frac{1}{3t} \right]
\]

\[
\frac{\partial^2 u}{\partial x^2} = -\frac{1}{2t^{3/2}} \left[ e^{-\frac{x^2}{4t}} \left( \frac{\partial}{\partial x} \left( \frac{x^2}{4t} \right) \right) + e^{-\frac{x^2}{4t}} \right]
\]

\[
= \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \left( \frac{x^2}{4t^2} - \frac{1}{3t} \right) = \frac{2u}{\sqrt{\pi t}}
\]

\[ u \to 0 \quad \text{as} \quad x \to \pm \infty \quad \text{as} \quad t \to 0 \quad \text{fixed} \]

As \( t \to 0+ \), the profile becomes more and more concentrated around \( x = 0 \).

They converge to the Dirac delta function.
2. Determine for each of the following functions whether it is periodic, whether it is piecewise continuous, and whether it is even, odd, or neither. In the case of periodic functions, also determine the fundamental period.

(a) $f(x) = \cos^3 x$

   periodic, piecewise continuous, even
   $T = 3\pi$

(b) $f(x) = \cos(2x^3)$

   non periodic, piecewise continuous, even

(c) $f(x) = (x^3 + \sin x)^2$

   non periodic, piecewise continuous, even

(d) $f(x) = \tan |x|$

   non periodic, not piecewise continuous, even
3. Find the Fourier series of the $2\pi$-periodic function plotted below.

$$a_0 = \frac{1}{2\pi} \left[ \int_{-\pi}^{0} (-1) \, dx + \int_{0}^{\pi} 2 \, dx \right] = \frac{1}{2}$$

$$a_n = \frac{1}{\pi} \left[ \int_{-\pi}^{0} (-1) \cos nx \, dx + \int_{0}^{\pi} 2 \cos nx \, dx \right] = \frac{1}{\pi} \left[ 0 + 0 \right] = 0$$

$$b_n = \frac{1}{\pi} \left[ \int_{-\pi}^{0} (-1) \sin nx \, dx + \int_{0}^{\pi} 2 \sin nx \, dx \right] = \frac{1}{\pi} \left[ \frac{1}{n} (1 - \cos n\pi) \right]$$

$$= \frac{3}{n\pi} (1 - \cos n\pi) = \begin{cases} 0 & n \text{ even} \\ \frac{6}{n\pi} & n \text{ odd} \end{cases}$$

$$f(x) = \frac{1}{2} + \frac{6}{\pi} \sum_{k=0}^{\infty} \frac{\sin (2k+1)x}{2k+1}$$
4. Find the cosine expansion of the function

\[ g(x) = \sin x, \quad 0 < x < \pi. \]

Comment on the relationship between this expansion and the Fourier series for the 2\pi-periodic function

\[ f(x) = |\sin x|, \quad -\pi < x < \pi. \]

The cosine expansion of \( g(x) \) should be exactly the same as the Fourier series for \( f(x) \):

\[
\frac{a_0}{\pi} = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k)^2 - 1} \cos 2kx
\]

\[ a_0 = \frac{1}{\pi} \int_0^\pi \sin x \, dx = -\frac{1}{\pi} \left[ \cos \pi - \cos 0 \right] = \frac{2}{\pi} \]

\[ a_n = \frac{2}{\pi} \int_0^\pi \sin x \cos nx \, dx = \frac{2}{\pi} \left[ \frac{\cos((n-1)x)}{x(n-1)} - \frac{\cos((n+1)x)}{2(n+1)} \right]_0^\pi \]

\[ = \frac{1}{\pi} \left[ \frac{\cos((n-1)\pi-1)}{n-1} - \frac{\cos((n+1)\pi-1)}{2(n+1)} \right] \quad \text{if} \quad n \neq 1 \]

\[
= \begin{cases} 
0 & n \text{ odd} \\
\frac{4}{\pi} \frac{1}{n-1} & n \text{ even}
\end{cases}
\]

\[ a_1 = \frac{2}{\pi} \int_0^\pi \sin x \cos x \, dx = 0 \]