Partial Differential Equations 3150
Sample Midterm Exam 1
Exam Date: Tuesday, 27 October 2009

Instructions: This exam is timed for 50 minutes. You will be given double time to complete the exam. No calculators, notes, tables or books. Problems use only chapters 1 and 2 of the textbook. No answer check is expected. Details count 3/4, answers count 1/4.

1. (Vibration of a Finite String)
   Some normal modes for the string equation \( u_{tt} = c^2 u_{xx} \) are given by the equation
   \[
   u(x, t) = \sin \left( \frac{n\pi x}{L} \right) \cos \left( \frac{n\pi ct}{L} \right).
   \]
   (a) [25\%] Give an example of a finite linear combination of normal modes.
   (b) [25\%] Write a mathematical argument, using the superposition principle, showing that the example given in (a) is a solution of \( u_{tt} = c^2 u_{xx} \).
   (c) [50\%] Solve the finite string vibration problem on \( 0 \leq x \leq 1, \ t > 0 \),

   \[
   \begin{align*}
   u_{tt} & = c^2 u_{xx}, \\
   u(0, t) & = 0, \\
   u(1, t) & = 0, \\
   u(x, 0) & = 2 \sin(\pi x) - 3 \sin(5\pi x), \\
   u_t(x, 0) & = 0.
   \end{align*}
   \]

Use this page to start your solution. Attach extra pages as needed, then staple.
2. (Periodic Functions)

(a) [25%] Find the period of \( f(x) = \sin 2x \cos 2x \).

(b) [25%] Give an example of a piecewise continuous function on \( 0 \leq x \leq 2 \) that has a discontinuity at \( x = 1 \).

(c) [25%] Is \( f(x) = \cos(2x + 3) \) an even periodic function?

(d) [25%] Is \( f(x) = \sin(\pi x/5) \) an odd periodic function?
3. (Fourier Series)
Let \( f(x) = 1 \) on the interval \( 0 < x < 2\pi \), \( f(x) = -1 \) on \( -2\pi < x < 0 \), \( f(x) = 0 \) for \( x = 0, 2\pi, -2\pi \). Let \( g(x) \) be the \( 4\pi \)-periodic extension of \( f \) to the whole real line.

(a) [25\%] Is \( g(x) \) even or odd?
(b) [25\%] Display the formulas for the Fourier coefficients of \( f \).
(c) [25\%] Compute the Fourier coefficient for the term \( \sin(5x) \).
(d) [25\%] Are there any values of \( x \) such that \( g(x) \) does not equal the Fourier series of \( f \)?
4. (Cosine and Sine Series)
Find the first three terms in the cosine series expansion of the cosine wave $g(x)$, formed as the even periodic extension of the base function $\cos x + 2 \cos 4x$ on $0 < x < \pi$.
5. (Convergence of Fourier Series)

(a) [25%] Display Dirichlet’s kernel formula.
(b) [25%] State the Fourier Convergence Theorem for piecewise smooth functions.
(c) [25%] Fourier convergence may not be uniform, and the commonly referenced term to describe this problem is Gibbs’s phenomenon. Explain what it is, by example.
(d) [25%] State Parseval’s identity for complex Fourier series.