Theorem 1 (First Order Recipe)
Let $a$ and $b$ be constants, $a \neq 0$. Let $r_1$ denote the root of $ar + b = 0$. Then $y = c_1 e^{r_1x}$ is the general solution of the first order equation

$$ ay' + by = 0. $$

Theorem 2 (Second Order Recipe)
Let $a \neq 0$, $b$ and $c$ be real constants. Let $r_1$, $r_2$ be the two roots of $ar^2 + br + c = 0$, real or complex. If complex, then let $r_1 = \overline{r_2} = \alpha + i\beta$ with $\beta > 0$. Consider the following three cases, organized by the sign of the discriminant $D = b^2 - 4ac$:

- $D > 0$ (Real distinct roots) $y_1 = e^{r_1x}$, $y_2 = e^{r_2x}$.
- $D = 0$ (Real equal roots) $y_1 = e^{r_1x}$, $y_2 = xe^{r_1x}$.
- $D < 0$ (Conjugate roots) $y_1 = e^{\alpha x} \cos(\beta x)$, $y_2 = e^{\alpha x} \sin(\beta x)$.

Then $y_1$, $y_2$ are two solutions of $ay'' + by' + cy = 0$ and the general solution is given by $y = c_1 y_1 + c_2 y_2$, where $c_1$, $c_2$ are arbitrary constants.
Theorem 3 (Picard-Lindelöf Existence-Uniqueness)
Let the coefficients $a(x), b(x), c(x), f(x)$ be continuous on an interval $J$ containing $x = x_0$. Assume $a(x) \neq 0$ on $J$. Let $y_0$ and $y_1$ be constants. The initial value problem

$$a(x)y'' + b(x)y' + c(x)y = f(x),$$
$$y(x_0) = y_0, \quad y'(x_0) = y_1$$

has a unique solution $y(x)$ defined on $J$.

Theorem 4 (Superposition)
The homogeneous equation $a(x)y'' + b(x)y' + c(x)y = 0$ has the superposition property:

If $y_1, y_2$ are solutions and $c_1, c_2$ are constants, then the combination $y(x) = c_1y_1(x) + c_2y_2(x)$ is a solution.
Theorem 5 (Homogeneous Structure)
The homogeneous equation \( a(x) y'' + b(x) y' + c(x) y = 0 \) has a general solution of the form

\[
y_h(x) = c_1 y_1(x) + c_2 y_2(x),
\]

where \( c_1, c_2 \) are arbitrary constants and \( y_1(x), y_2(x) \) are solutions.

Theorem 6 (Non-Homogeneous Structure)
The non-homogeneous equation \( a(x) y'' + b(x) y' + c(x) y = f(x) \) has general solution

\[
y(x) = y_h(x) + y_p(x),
\]

where

- \( y_h(x) \) is the general solution of the homogeneous equation \( a(x) y'' + b(x) y' + c(x) y = 0 \), and
- \( y_p(x) \) is a particular solution of the nonhomogeneous equation \( a(x) y'' + b(x) y' + c(x) y = f(x) \).