Ch X: A Hopf Bifurcation Theorem

This chapter is devoted to a version of the classical **Hopf bifurcation** theorem which establishes the existence of nontrivial periodic orbits of autonomous systems of differential equations which depend upon a parameter and for which the stability properties of the trivial solution changes as the parameter is varied.

The proof is based on the **method of Lyapunov-Schmidt** presented in Chapter II.
Theorem (Hopf Bifurcation). Assume that $f$ satisfies the following conditions:

(1) $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a $C^2$ mapping such that $f(0, \alpha) = 0, \alpha \in \mathbb{R}$.

(2) For some given value of $\alpha = \alpha_0$, $i = \sqrt{-1}$ and $-i$ are eigenvalues of $f_u(0, \alpha_0)$ and $\pm ni, n = 0, 2, 3, \cdots$ are not eigenvalues of $f_u(0, \alpha_0)$;

(3) in a neighborhood of $\alpha_0$ there is a curve of eigenvalues and eigenvectors

$$f_u(0, \alpha)a(\alpha) = \beta(\alpha)a(\alpha)$$

$$a(\alpha_0) \neq 0, \beta(\alpha_0) = i, \text{Re} \frac{d\beta}{d\alpha}|_{\alpha_0} \neq 0.$$  

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Hopf Bifurcation – continued

Then there exist positive numbers $\epsilon$ and $\eta$ and a $C^1$ function $(u, \rho, \alpha) : (-\eta, \eta) \to C_{2\pi}^1 \times \mathbb{R} \times \mathbb{R}$, where $C_{2\pi}^1$ is the space of $2\pi$ periodic $C^1 \mathbb{R}^n$–valued functions, such that $(u(s), \rho(s), \alpha(s))$ solves the equation $\frac{du}{d\tau} + \rho f(u, \alpha) = 0$, nontrivially, i.e., $u(s) \neq 0$, $s \neq 0$ and $\rho(0) = 1$, $\alpha(0) = \alpha_0$, $u(0) = 0$.

Furthermore, if $(u_1, \alpha_1)$ is a nontrivial solution of $\frac{du}{dt} + f(u, \alpha) = 0$ of period $2\pi \rho_1$, with $|\rho_1 - 1| < \epsilon$, $|\alpha_1 - \alpha_0| < \epsilon$, $|u_1(t)| < \epsilon$, $t \in [0, 2\pi \rho_1]$, then there exists $s \in (-\eta, \eta)$ such that $\rho_1 = \rho(s)$, $\alpha_1 = \alpha(s)$ and $u_1(\rho_1 t) = u(s)(\tau + \theta)$, $\tau = \rho_1 t \in [0, 2\pi]$, $\theta \in [0, 2\pi)$.
Van der Pol Oscillator Example

Consider the nonlinear oscillator

\[ x'' + x - \alpha(1 - x^2)x' = 0. \]

This equation will be shown to have for certain small values of \( \alpha \) nontrivial periodic solutions with periods close to \( 2\pi \).

\textbf{Details.} Transform the equation into a system via \( u_1 = x, \ u_2 = x', \ u = (u_1, u_2)^T \) to obtain

\[ u' + \begin{pmatrix} 0 & -1 \\ 1 & -\alpha \end{pmatrix} u + \begin{pmatrix} 0 \\ u_1^2 u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

Then \( f(u, \alpha) = \begin{pmatrix} 0 & -1 \\ 1 & -\alpha \end{pmatrix} u + \begin{pmatrix} 0 \\ u_1^2 u_2 \end{pmatrix} \) and

\[ f_u(0, \alpha) = \begin{pmatrix} 0 & -1 \\ 1 & -\alpha \end{pmatrix}. \]

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Details continued

The eigenvalues of $f_u(0, \alpha)$ satisfy $\beta(\alpha + \beta) + 1 = 0$. Let $\alpha_0 = 0$, then $\beta(0) = \pm i$, and computing $\frac{d\beta}{d\alpha} = \beta'$ we obtain $2\beta\beta' + \beta'\alpha + \beta = 0$, or $\beta' = \frac{-\beta}{\alpha + 2\beta} = -\frac{1}{2}$, for $\alpha = 0$. Thus by Theorem II of chapter III there exists $\eta > 0$ and continuous functions $\alpha(s)$, $\rho(s)$, $s \in (-\eta, \eta)$ such that $\alpha(0) = 0$, $\rho(0) = 1$ and the differential equation has for $s \neq 0$ a nontrivial solution $x(s)$ with period $2\pi\rho(s)$. This solution is unique up to phase shift.