Chapter VI: Linear Equations

Let $I$ be a real interval. Let $A : I \rightarrow \mathbf{L}(\mathbb{R}^{N}, \mathbb{R}^{N})$ and $f : I \rightarrow \mathbb{R}^{N}$ be continuous functions. Consider the linear systems $u' = A(t)u + f(t), \ t \in I,$ and $u' = A(t)u, \ t \in I.$

**Proposition 1.** The initial value problem $u' = A(t)u + f(t), \ u(t_0) = u_0$ is uniquely solvable for each $t_0 \in I, \ u_0 \in \mathbb{R}^{N}$ and the solution $u(t)$ is defined on all of $I.$

If $A$ and $f$ are measurable on $I$ and locally integrable there, then a parallel theory can be developed.

**Proposition 2.** The set of solutions of $u' = A(t)u$ is a vector space of dimension $N.$


**Fundamental Solutions**

**Lemma 4 (Abel–Liouville).** Let $\Phi(t)$ be an $N \times N$ matrix solution of $u' = A(t)u$. Then $g(t) = \det \Phi(t)$ satisfies the differential equation $g' = \text{trace}(A(t))g$. In particular, $\Phi(t)$ is nonsingular for all $t \in I$ if and only if $\Phi(t_0)$ is nonsingular for one $t_0 \in I$.

**Definition.** A nonsingular matrix $\Phi(t)$ whose columns are solutions of $u' = A(t)u$ is called a **fundamental matrix solution** or a fundamental system.

**Proposition 5.** Let $\Phi$ be a given fundamental matrix solution of $u' = A(t)u$. Then every other fundamental matrix solution $\Psi$ has the form $\Psi = \Phi C$, where $C$ is a constant nonsingular $N \times N$ matrix. Furthermore the set of all solutions of $u' = A(t)u$ is given by $\{\Phi c : c \in \mathbb{R}^N\}$, where $\Phi$ is a fundamental system.
Variation of Constants Formula

**Proposition 6.** Let $\Phi$ be a fundamental matrix solution of $u' = A(t)u$ and let $t_0 \in I$. Then $u_p(t) = \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)f(s)ds$ is a solution of $u' = A(t)u + f(t)$. Hence the set of all solutions of $u' = A(t)u + f(t)$ is given by

$$\left\{ \Phi(t) \left( c + \int_{t_0}^{t} \Phi^{-1}(s)f(s)ds \right) : c \in \mathbb{R}^N \right\},$$

where $\Phi$ is a fundamental system of $u' = A(t)u$.

**Exponential Matrix**

$$e^{At} = \sum_{n=0}^{\infty} A^n \frac{t^n}{n!}.$$ 

This series expansion is valid for constant matrices $A$. It converges uniformly on compact $t$-sets. The series represents a fundamental matrix for the equation $u' = Au$ which is the identity matrix at $t = 0$. 

78
Real Jordan Form

The matrix formula $J = P^{-1}AP$ summarizes the real Jordan form of $A$. In this form, $P$ is formed from the real and imaginary parts of the generalized eigenvectors of $A$, while $J = \text{diag}(J_1, \ldots, J_k)$; the matrices $J_1, \ldots, J_k$ are called **Jordan blocks**. The structure of a Jordan block is as follows: the diagonal entries are either a real eigenvalue $\lambda$ of $A$ or else the $2 \times 2$ matrix $\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$, which corresponds to the complex eigenvalue $\alpha + i\beta$. On the super-diagonal of the Jordan block there are ones (1) or $2 \times 2$ identity matrices.
Calculation of $e^{AT}$

If matrices $E$ and $N$ commute, then $e^{E+N} = e^Ee^N$. A Jordan block $C$ can be written as a sum $C = E + N$ where $E$ is block-diagonal, $N$ is nilpotent ($N^r = 0$ for some $r \geq 1$) and $EN = NE$. Therefore, $e^{Ct} = e^{ET}e^{Nt}$. The exponential $e^{Et}$ is again block-diagonal, while the series $e^{Nt}$ is a finite sum. There are two cases, corresponding to real or complex eigenvalues of $A$.

Proposition 7. Let $A$ be an $N \times N$ constant matrix and consider the differential equation $u' = Au$. Then:
1. All solutions $u$ of $u' = Au$ satisfy $u(t) \to 0$, as $t \to \infty$, if and only if $\text{Re}\lambda < 0$, for all eigenvalues $\lambda$ of $A$.
2. All solutions $u$ of $u' = Au$ are bounded on $[0, \infty)$, if and only if $\text{Re}\lambda \leq 0$, for all eigenvalues $\lambda$ of $A$ and those with zero real part are semisimple.
Floquet Theory

Let $A(t)$ be an $N \times N$ continuous matrix such that $A(t + T) = A(t)$, $-\infty < t < \infty$. Consider the differential equation $u' = A(t)u$.

**Proposition 8.** Let $\Phi(t)$ be a fundamental matrix solution of $u' = A(t)u$ with $A(t)$ $T$-periodic and continuous. Then $\Psi(t) = \Phi(t + T)$ is also a fundamental matrix.

**Theorem 9 (Floquet).** Let the $N \times N$ matrix $A(t)$ be $T$-periodic and continuous. There exists a constant matrix $R$ and a $T$-periodic nonsingular matrix $C(t)$ such that the change of variable $u = C(t)y$ changes $u' = A(t)u$ into the constant-coefficient equation $y' = R y$.

In particular, there is a fundamental matrix $\Phi(t)$ for $u' = A(t)u$ of the form $\Phi(t) = C(t)e^{Rt}$, where $R$ is a constant matrix and $C(t)$ is a $T$-periodic nonsingular matrix of class $C^1$. If $\Phi(0) = I$, then $C(0) = C(T) = I$ and $e^{RT} = \Phi(T)$.
Corollary 10. Let the $N \times N$ matrix $A(t)$ be $T$-periodic and continuous. Let $\Phi(t)$ be a fundamental matrix for $u' = A(t)u$. Then, there exists a solution $u(t) \neq 0$ of period $mT$ if and only if $\Phi^{-1}(0)\Phi(T)$ has an eigenvalue $\lambda$ with $\lambda^m = 1$.

Solving $Q = e^X$ for $X$ when $\det(Q) \neq 0$

The condition $\det(Q) \neq 0$ means that all eigenvalues of $Q$ are nonzero. Write $Q = P^{-1}JP$ where $J$ is a block-diagonal matrix whose diagonal entries are complex Jordan blocks.

It suffices to solve for $X$ in $\lambda I + N = e^X$ when $\lambda \neq 0$, $I$ is the identity and $N$ is nilpotent, because this is the form of a complex Jordan block.

A candidate for the solution $X$ in this special case is given by a formal logarithmic series $X = \lambda \ln(1 + N/\lambda) = \ln(\lambda)I - \sum_{k=1}^{P} (-N/\lambda)^k/k$ where $N^P = 0$. To verify that this solution $X$ indeed satisfies $\lambda I + N = e^X$ is routine, because there is no issue of convergence.
Hill’s equation $y'' + p(t)y = 0$

Write $y'' + p(t)y = 0$ as a system $u' = A(t)u$ where

$$A = \begin{bmatrix} 0 & 1 \\ -p(t) & 0 \end{bmatrix}, \quad u = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}.$$  

Let $\Phi(t)$ be a fundamental matrix with $\Phi(0) = I$. Let $[y_1, y_2]$ be the first row of $\Phi$ and define $a = y_1(T) + y_2'(T)$. Corollary 10 says that Hill’s equation has a periodic solution of period $mT$ if and only if $\Phi(T)$ has an eigenvalue $\lambda$ with $\lambda^m = 1$. An eigenvalue $\lambda$ must satisfy $\lambda^2 - a\lambda + 1 = 0$, therefore the condition for an $mT$-periodic solution is

$$\left( \frac{a \pm \sqrt{a^2 - 4}}{2} \right)^m = 1.$$