Mathematics 5410
Phase Diagrams

Example. Plot the phase diagram of the linear dynamical system

\[ x' = -2x + 3y, \quad y' = -3x - 2y. \]

Solution: The Maple V.4 code which does the plot appears below.

```maple
code:=
restart:with(DEtools):
h:=0.2:k:=0.2:
de:=[diff(x(t),t)=-2*x(t)+3*y(t),
    diff(y(t),t)=-3*x(t)-2*y(t)]:
vars:=[x(t),y(t)]:
inits:=[seq(seq([0,i*h,j*k],i=-2..2),j=-2..2)]:
dsolve(de,vars);
phaseportrait(de,vars,t=0..2*Pi,inits);
```

Problem 1. Plot a phase diagram for the matrix systems \( X' = AX \),
given matrix \( A \) below, and classify as center, stable spiral, unstable spiral, proper node, improper node, saddle. Subdivide the proper nodes into the two cases deficient node and star node, as in Borrelli–Coleman, page 389.

\[
A_1 = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix},
\]

\[
A_4 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_5 = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_6 = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix},
\]

\[
A_7 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \quad A_8 = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}, \quad A_9 = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix},
\]

\[
A_{10} = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}.
\]

Problem 2. Reproduce the soft spring phase diagram for (see Borrelli–Coleman page 153)

\[ x' = y, \quad y' = -10x + 0.2x^3 - 0.2y - 9.8. \]

Classify the equilibria (see Borrelli–Coleman page 389)

\[
(-1,0), \quad ((1 + \sqrt{197})/2,0), \quad ((1 - \sqrt{197})/2,0).
\]

Hint: Use after the \texttt{inits} entry, the options
to make a finer plot without the direction field. Read the Maple help on `phaseportrait`.

There are about 12 significant initial conditions to plot, based upon Figure 3.1.2, page 153. Choose the three equilibria for a start, then add others until the figure matches the one in the text.

**Problem 3.** Consider the soft spring phase diagram for (see Borrelli–Coleman page 153)

\[
\begin{align*}
x' &= y, \\
y' &= -10x + 0.2x^3 - 0.2y - 9.8.
\end{align*}
\]

Make three individual phase diagrams for each of the three linearized equations about the equilibrium points

\[-1, 0), \quad (1 + \sqrt{197})/2, 0), \quad (1 - \sqrt{197})/2, 0).
\]

The linearized differential equations about these equilibria are (see Borrelli–Coleman, page 154, for the first one):

\[
\begin{align*}
x' &= y, \quad y' = 9.4(x + 1) - 0.2y, \\
x' &= y, \quad y' = (197 + 3\sqrt{197})(x - 1/2 - \sqrt{197}/2)/10 - 0.2y, \\
x' &= y, \quad y' = (197 - 3\sqrt{197})(x - 1/2 + \sqrt{197}/2)/10 - 0.2y.
\end{align*}
\]

**Problem 4.** Justify mathematically the linearized differential equations given in Problem 3. This work is to be handwritten and full of detail.

**Taylor’s Theorem.** Let \( F(X) \) be a function from \( \mathbb{R}^n \) into \( \mathbb{R}^n \), twice continuously differentiable. Let \( X_0 \) be a given point. Then

\[
F(X) = F(x_0) + J(X - X_0) + \mathcal{R}
\]

where \( J \) is the Jacobian matrix of \( F \) at \( X = X_0 \) and the remainder \( \mathcal{R} \) satisfies \( |\mathcal{R}| \leq K|X - X_0|^2 \) as \( X \) approaches \( X_0 \).

The Jacobian matrix has entries \( \partial F_i / \partial x_j \), that is, the columns of \( J \) are the vector partials \( \partial x_i F(X_0) \).

**Linearized Equation.** At an equilibrium point \( X_0 \), the dynamical system \( X' = F(X) \) has linearization \( X' = J(X - X_0) \). This equation is formally obtained from its nonlinear counterpart \( X' = F(X) \) by dropping the Taylor remainder and observing that, by assumption, \( F(X_0) = 0 \).
Example. Compute the Jacobian matrix at \( X_0 = (-1, 0) \) for the vector function
\[
F(x, y) = \begin{pmatrix}
y \\
-10x + 0.2x^3 - 0.2y - 9.8
\end{pmatrix}
\]
and find the linearized system for \( X' = F(X) \).

Solution: The partials are
\[
\begin{align*}
\partial_x F &= \partial_x \begin{pmatrix}
y \\
-10x + 0.2x^3 - 0.2y - 9.8
\end{pmatrix} \\
&= \begin{pmatrix}
0 \\
-10 + 0.6x^2
\end{pmatrix} \\
&= \begin{pmatrix}
0 \\
-10 + 0.6(-1)^2
\end{pmatrix} \\
&= \begin{pmatrix}
0 \\
-9.4
\end{pmatrix}, \\
\partial_y F &= \partial_y \begin{pmatrix}
y \\
-10x + 0.2x^3 - 0.2y - 9.8
\end{pmatrix} \\
&= \begin{pmatrix}
1 \\
-0.2
\end{pmatrix},
\end{align*}
\]

\[
J = \begin{pmatrix}
0 & 1 \\
-9.4 & -0.2
\end{pmatrix}.
\]
The linearized system is \( X' = J(X - X_0) \), or
\[
\begin{pmatrix}
x' \\
y'
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
-9.4 & -0.2
\end{pmatrix} \begin{pmatrix}
x + 1 \\
y
\end{pmatrix}.
\]