Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected.

1. (Variation of parameters)
   (a) [50%] State and prove the variation of parameters formula for a second order linear differential equation.
   (b) [50%] Solve by variation of parameters \( y'' - y = xe^x \).
   (c) [50%] Solve by variation of parameters \( u' = Au + F(t) \), given \( A = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} \) and \( F(t) = \begin{pmatrix} t \\ 1 + t \end{pmatrix} \). The answer contains two arbitrary constants \( c_1, c_2 \).

2. ()
   (a) [50%] Find the linearized equation at each equilibrium point: \( x' = xy - y, y' = x^2 - x^3 y \).
   (b) [50%] Classify the equilibria as stable or unstable: \( x' = 3x, y' = x(y - 1), z' = x + z \).
   (c) [50%] Prove that a matrix equation \( x' = Ax \) is asymptotically stable at \( x = 0 \), if the real part of each eigenvalue of \( A \) is negative.
   (d) [50%] Classify as a center, spiral, saddle or node: \( x' = Ax, A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \), when \( b > 0 \), for all possible choices of \( a \).

3. (Linear 3 \times 3 systems)
   (a) [50%] Solve \( x' = x - 2y, y' = x + y, z' = z \) by Putzer’s spectral recipe.
   (b) [50%] Solve \( x' = x - z, y' = y - x, z' = z + y \) by eigenanalysis.
   (c) [50%] Solve \( x' = x - z, y' = y - x, z' = z + y \) by a spectral formula.

4. (Resonance)
   (a) [50%] Define pure resonance. Find a periodic solution in the non-resonant case for \( x'' + 16x = \cos(\omega t) \).
   (b) [50%] Define practical resonance. Find the unique periodic solution of \( x'' + 2x' + x = \cos(\omega t) \).
   (c) [50%] Prove that practical resonance occurs exactly when \( \omega = \sqrt{k/m - c^2/(2m^2)} \) is positive, for the equation \( mx'' + cx' + kx = F_0 \cos(\omega t) \).

5. (Theory of linear systems)
   (a) [50%] State the existence-uniqueness theorem for \( x' = A(t)x + F(t) \). Include a statement about the domain of the unique solution \( x(t) \).
   (b) [50%] State and prove the superposition principle for \( x' = Ax + f(t) \).
   (c) State and prove Abel’s formula for \( x' = Ax \).