4.4 Undetermined Coefficients

The **method of undetermined coefficients** applies to solve differential equations

\[ ay'' + by' + cy = f(x). \]  

**Restrictions:** The symbols \( a, b, c \) are constant, \( a \neq 0 \). The nonhomogeneous term \( f(x) \) is a sum of terms, each of which is one the following forms, called **atoms**: 

\[
\begin{align*}
  p(x) & \quad \text{polynomial,} \\
  p(x)e^{kx} & \quad \text{polynomial} \times \text{exponential}, \\
  p(x)e^{kx}\cos(mx) & \quad \text{polynomial} \times \text{exponential} \times \text{cosine}, \\
  p(x)e^{kx}\sin(mx) & \quad \text{polynomial} \times \text{exponential} \times \text{sine}.
\end{align*}
\]

The polynomial \( p(x) \) can be a constant. Symbols \( k \) and \( m \geq 0 \) are constants. The trigonometric terms may appear without an exponential, e.g., \((1 + 2x)e^{0x}\sin 3x\) is normally written \((1 + 2x)\sin 3x\). The method’s importance is argued from its direct applicability to second order differential equations in mechanics and circuit theory.

**Included** as possible functions \( f \) in (1) are \( \sinh x \) and \( \cos^3 x \), due to identities from algebra and trigonometry. **Specifically excluded** are \( \ln |x|, |x|, e^{x^2} \) and fractions like \( x/(1 + x^2) \).

**Superposition** \( y = y_h + y_p \) allows us to solve equation (1) in two stages: (a) Apply the linear equation **recipe** to find \( y_h \); (b) Apply the **method of undetermined coefficients** to find \( y_p \). We expect to find two arbitrary constants \( c_1, c_2 \) in the solution \( y_h \), but in contrast, no arbitrary constants appear in \( y_p \).

The **basic trial solution method**, which requires linear algebra, is presented on page 174. Readers should make an effort to learn this method, because literature normally omits details of the method, referencing only the **method of undetermined coefficients**. To enrich this basic method, we add a **library of special methods** for finding \( y_p \), which includes Kummer’s method; see page ???. The library uses only college algebra and polynomial calculus. The trademark of the library method is the **absence of linear algebra, tables or special cases**, that can be found in other literature on the subject.

**The Algorithm for Undetermined Coefficients**

A particular solution \( y_p \) of (1) will be expressed as a sum

\[ y_p = y_1 + \cdots + y_n \]

where each \( y_k \) solves a related easily-solved differential equation.
4.4 Undetermined Coefficients

The idea can be quickly communicated for \( n = 3 \). The superposition principle applied to the three equations

\[
\begin{align*}
ay''_1 + by'_1 + cy_1 &= f_1(x), \\
ay''_2 + by'_2 + cy_2 &= f_2(x), \\
ay''_3 + by'_3 + cy_3 &= f_3(x)
\end{align*}
\]

(3)

shows that \( y = y_1 + y_2 + y_3 \) is a solution of

\[
ay'' + by' + cy = f_1 + f_2 + f_3.
\]

(4)

If each equation in (3) is easily solved, then solving equation (4) is also easy: add the three answers for the easily solved problems.

To use the idea, it is necessary to start with \( f(x) \) and determine a decomposition \( f = f_1 + f_2 + f_3 \) so that equations (3) are easily solved.

The process is called the method of undetermined coefficients. This method requires decomposing (1) into a number of easily-solved equations. For instance, if an easily-solved equation has forcing term \( f(x) \) equal to a polynomial, then a particular solution is found by substituting a polynomial trial solution

\[
y = d_0 + d_1 x + \cdots + d_m \frac{x^m}{m!}
\]

with undetermined coefficients \( d_0, \ldots, d_m \). Undetermined coefficients are found by calculus and college algebra back-substitution.

The Easily Solved Equations. Each easily-solved equation is engineered to have right side in one of the four forms below, each of which is called an atom:

\[
\begin{align*}
p(x) & \quad \text{polynomial}, \\
p(x)e^{kx} & \quad \text{polynomial} \times \text{exponential}, \\
p(x)e^{kx}\cos mx & \quad \text{polynomial} \times \text{exponential} \times \text{cosine}, \\
p(x)e^{kx}\sin mx & \quad \text{polynomial} \times \text{exponential} \times \text{sine}.
\end{align*}
\]

(5)

To illustrate, consider

\[
ay'' + by' + cy = x + xe^x + x^2 \sin x - \pi e^{2x} \cos x + x^3.
\]

(6)

The right side is decomposed as follows, in order to define the easily solved equations (also called the atomic equations):

\[
\begin{align*}
ay''_1 + by'_1 + cy_1 &= x + x^3 & \quad \text{Polynomial}. \\
ay''_2 + by'_2 + cy_2 &= xe^x & \quad \text{Polynomial} \times \text{exponential}. \\
ay''_3 + by'_3 + cy_3 &= x^2 \sin x & \quad \text{Polynomial} \times \text{exponential} \times \text{sine}. \\
ay''_4 + by'_4 + cy_4 &= -\pi e^{2x} \cos x & \quad \text{Polynomial} \times \text{exponential} \times \text{cosine}.
\end{align*}
\]

There are \( n = 4 \) equations. In the illustration, \( x^3 \) is included with \( x \), but it could have caused creation of a fifth equation. To decrease effort, minimize the number \( n \) of easily solved equations. One final checkpoint: the right sides of the \( n \) equations must add to the right side of (6).
The Basic Trial Solution Method

Literature referencing the method of undetermined coefficients usually means the basic trial solution method. Readers are asked to spend enough time to understand the method’s mechanics and intricacies.

Assume given a constant-coefficient second order differential equation \( ay'' + by' + cy = f(x) \), with \( f \) an atom, as in (2). The method:

**Homogeneous solution.** Solve the homogeneous equation for \( y_h \) by the recipe. It contains arbitrary constants \( c_1, c_2 \).

**Initial trial solution.** Differentiate the atom \( f(x) \) repeatedly. Isolate independent functions whose linear combinations are the derivatives. Multiply them by undetermined coefficients \( d_1, d_2, \ldots, d_k \) to define an initial trial solution.

**Fixup rule.** If the initial trial solution duplicates terms found in \( y_h \), then multiply the trial solution by \( x \) repeatedly until it doesn’t. The final trial solution is the modified expression.

**Substitute and evaluate.** Substitute the final trial solution into the nonhomogeneous differential equation. Match coefficients of the independent functions to obtain equations for the undetermined coefficients \( d_1, d_2, \ldots, d_k \). Solve the system.

**Report** \( y = y_h + y_p \). Homogeneous solution \( y_h \) was reported above. Particular solution \( y_p \) is the final trial solution with the evaluated coefficients. Add to obtain the general solution \( y \).

The algorithm actually works for sums of atoms, provided the fixup rule is applied to each individual atom. Linear algebra techniques are used to solve the system of equations in the 4th step. An answer check is prudent, because of many opportunities for arithmetic errors.

**Illustration.** Let’s solve \( y'' - y = x + xe^x \), verifying that \( y_h = c_1e^x + c_2e^{-x} \) and \( y_p = -x - \frac{1}{4}xe^x + \frac{1}{4}x^2e^x \).

- **Homogeneous solution.** The characteristic equation \( r^2 - 1 = 0 \) has roots \( r = \pm 1 \). Recipe case 1 implies \( y_h = c_1e^x + c_2e^{-x} \).

- **Initial trial solution.** The atoms of \( f = x + xe^x \) are \( f_1 = x, f_2 = xe^x \). Then \( f = f_1 + f_2 \) and the easily-solved problems are \( y_1'' - y_1 = x \) and \( y_2'' - y_2 = xe^x \). A particular solution is \( y_p = y_1 + y_2 \). Initial trial solutions, found by differentiation, involve the independent terms \( 1, x \) for \( y_1 \) and \( e^x, xe^x \) for \( y_2 \). Then \( y_1 = d_1 + d_2x, y_2 = d_3e^x + d_4xe^x \). The undetermined coefficients are \( d_1, d_2, d_3, d_4 \).
4.4 Undetermined Coefficients

- **Fixup rule.** No terms of \( y_1 \) match those of \( y_h \), so \( y_1 \) is the final trial solution for \( y''_1 - y_1 = x \). Terms of \( y_2 \) match in \( y_h \). Multiplication once by \( x \) is required to eliminate duplicates. Then \( y_2 = x(d_3 e^x + d_4 x e^x) \) is the final trial solution.

- **Substitute and evaluate.** The details for \( y_1 \):

  \[
  x = y''_1 - y_1 \quad \text{Reverse sides in the equation.}
  
  = 0 - (d_1 + 2d_2 x) \quad \text{Substitute } y_1 = d_1 + 2d_2 x.
  
  = (-d_1) + (-2d_2) x \quad \text{Collect on powers of } x.
  
  \]

  Equating matching powers in the last equation gives the system of equations
  
  \[
  0 = -d_1, \\
  1 = -d_2.
  \]

  Therefore, \( d_1 = 0, \ d_2 = -1 \) and \( y_1 = -x \).

- **Substitute and evaluate.** The details for \( y_2 \):

  \[
  x e^x = y''_2 - y_2 \quad \text{Reverse equation sides.}
  
  = (d_3 x e^x + d_4 x^2 e^x)' \\
  \hspace{1cm} - (d_3 x e^x + d_4 x^2 e^x) \quad \text{Use } y_2 = d_3 x e^x + d_4 x^2 e^x.
  
  = 2d_4 e^x + 2d_3 x e^x + 4d_4 x e^x \quad \text{Differentiate and simplify.}
  
  \]

  Matching like terms left and right gives the system of equations
  
  \[
  0 = 2d_3 + 2d_4, \\
  1 = 4d_4.
  \]

  Then \( d_4 = 1/4, \ d_3 = -1/4 \) and \( y_2 = -\frac{1}{4} x e^x + \frac{1}{4} x^2 e^x \).

- **Report** \( y = y_h + y_p \). From above, \( y_h = c_1 e^x + c_2 e^{-x} \) and \( y_p = y_1 + y_2 = -x - \frac{1}{4} x e^x + \frac{1}{4} x^2 e^x \). Then \( y = y_h + y_p \) is given by

  \[
  y = c_1 e^x + c_2 e^{-x} - x - \frac{1}{4} x e^x + \frac{1}{4} x^2 e^x.
  \]

  **Answer check.** Computer algebra system **maple** is used.

```maple
yh:=c1*exp(x)+c2*exp(-x);
y1:=-x;
y2:=-(1/4)*x*exp(x)+(1/4)*x^2*exp(x);
d:=diff(y(x),x,x)-y(x)=x*exp(x);
odetest(y(x)=yh+y1+y2,d);
```

5 **Example (Sine–Cosine Trial solution)** Verify for \( y'' + 4y = \sin x - \cos x \) that \( y_p(x) = 5 \cos x + 3 \sin x \), by using the trial solution \( y = A \cos x + B \sin x \).
Solution: Substitute \( y = A \cos x + B \sin x \) into the differential equation and use \( u'' = -u \) for \( u = \sin x \) or \( u = \cos x \) to obtain the relation
\[
\sin x - \cos x = y'' + 4y = (-A + 4) \cos x + (-B + 4) \sin x.
\]
Comparing sides, matching sine and cosine terms, gives
\[
-A + 4 = -1, \quad -B + 4 = 1.
\]
Solving, \( A = 5 \) and \( B = 3 \). The trial solution \( y = A \cos x + B \sin x \) becomes \( y_p(x) = 5 \cos x + 3 \sin x \). Generally, this method produces linear algebraic equations that must be solved by linear algebra techniques.

6 Example (Basic Trial Solution Method: I)
Solve for \( y_p \) in \( y'' = 2 - x + x^3 \) using the basic trial solution method, verifying \( y_p = x^2 - x^3/6 + x^5/20 \).

Solution:
Homogeneous solution. The equation \( y'' = 0 \) has characteristic equation \( r^2 = 0 \) and therefore \( y_h = c_1 + c_2 x \).

Initial trial solution. The various derivatives of \( f(x) = 2 - x + x^3 \) are linear combinations of the independent terms \( 1, \ x, \ x^2, \ x^3 \). Then the initial trial solution is \( y = d_1 + d_2 x + d_3 x^2 + d_4 x^3 \).

Fixup rule and final trial solution. The homogeneous solution \( y_h = c_1 + c_2 x \) duplicates terms \( d_1 \) and \( d_2 x \) in the initial trial solution. Multiply \( y \) by \( x \) two times to eliminate duplications. Then \( y = x^2(d_1 + d_2 x + d_3 x^2 + d_4 x^3) \) is the final trial solution.

Substitute and evaluate. The details:
\[
2 - x + x^3 = y'' \quad \text{Reverse sides.}
\]
\[
= 2d_1 + 6d_2 x + 12d_3 x^2 + 20d_4 x^3 \quad \text{Substitute the final trial solution.}
\]
Equate like terms on each side of the equal sign to obtain the system of equations
\[
2d_1 = 2, \\
6d_2 = -1, \\
12d_3 = 0, \\
20d_4 = 1.
\]
This is a triangular system of linear equations for unknowns \( d_1, \ d_2, \ d_3, \ d_4 \). Solving gives \( d_1 = 1, \ d_2 = -1/6, \ d_3 = 0, \ d_4 = 1/20 \).

Report \( y_p \). The expression \( y = x^2(d_1 + d_2 x + d_3 x^2 + d_4 x^3) \) after substitution of the values found gives
\[
y = x^2(1 - x/6 + x^3/20).
\]

7 Example (Basic Trial Solution Method: II)
Solve \( y'' - y' + y = 2 + e^x + \sin(x) \) by the basic trial solution method, verifying \( y = c_1 e^{x/2} \cos(\sqrt{3}x/2) + c_2 e^{x/2} \sin(\sqrt{3}x/2) + 2 + e^x + \cos(x) \).
Solution:

Summary. There are three atoms: 2, $e^x$ and $\sin x$. The easily solved equations
are $y_1'' - y_1' + y_1 = 2$, $y_2'' - y_2' + y_2 = e^x$ and $y_3'' - y_3' + y_3 = \sin(x)$. Each
such equation is solvable by trial solution methods, giving $y_1 = 2$, $y_2 = e^x$ and
$y_3 = \cos x$. Then $y_p = y_1 + y_2 + y_3$ is the particular solution $y_p = 2 + e^x + \cos(x)$.

Homogeneous solution. The characteristic equation $r^2 - r + 1 = 0$ has roots
$(1 \pm i\sqrt{3})/2$. The recipe implies $y_h = (c_1 \cos \sqrt{3}x/2 + c_2 \sin \sqrt{3}x/2)e^{x/2}$, where
$c_1$ and $c_2$ are arbitrary constants.

Equation 1: $y_1'' - y_1' + y_1 = 2$. An initial trial solution is $y_1 = d_1 \cdot 1$, because
1 is the only independent function obtained by differentiation of the RHS. The
fixup rule changes nothing, due to no duplications in $y_h$. Then $y_1 = d_1$ is the
final trial solution. Substitution into $y_1'' - y_1' + y_1 = 2$ gives $d_1 = 2$. Then
$y_1 = 2$.

Equation 2: $y_2'' - y_2' + y_2 = e^x$. Differentiation of the RHS gives one indepen-
dent function $e^x$. Then $y_2 = d_2 e^x$ is the initial trial solution. The fixup rule
changes nothing, due to no duplications. Then $y_2 = d_2 e^x$ is the corrected trial
solution. Substitution into $y_2'' - y_2' + y_2 = e^x$ gives $(d_2 - d_2 + d_2)e^x = e^x$. Hence
d_2 = 1. Then $y_2 = e^x$.

Equation 3: $y_3'' - y_3' + y_3 = \sin(x)$. Differentiation of the RHS gives independent
functions $\cos x, \sin x$. The initial trial solution is $y_3 = d_3 \cos x + d_4 \sin x$. No
terms of $y_h$ are duplicated in $y_3$, therefore the fixup rule implies $y_3$ is the final
trial solution. Substitution into $y_3'' - y_3' + y_3 = \sin x$ gives $-d_3 \cos x - d_4 \sin x -$
$(-d_3 \sin x + d_4 \cos x) + (d_3 \cos x + d_4 \sin x) = \sin x$. Matching sine and cosine
terms left and right implies $-d_3 = 0, d_4 = 1$. Then $y_3 = \cos x$.

Solution $y_p$. The particular solution is given by addition, $y_p = y_1 + y_2 + y_3$,
with result $y_p = 2 + e^x + \cos(x)$.

General Solution. Add $y_h$ and $y_p$ to obtain the general solution

$$y = c_1 e^{x/2} \cos(\sqrt{3}x/2) + c_2 e^{x/2} \sin(\sqrt{3}x/2) + 2 + e^x + \cos(x).$$

8 Example (Basic Trial Solution Method: III)

Solve for $y_p$ in $y'' - 2y' + y = (1 + x - x^2)e^x$ by the basic trial solution
method, verifying that $y_p = (x^2/2 + x^3/6 - x^4/12)e^x$.

Solution: The right side is an atom, so there is no need to decompose it into
easily-solved problems.

Homogeneous solution. The characteristic equation $r^2 - 2r + 1 = 0$ for
$y'' - 2y' + y = 0$ has roots $r = 1, r = 1$. The recipe implies $y_h = c_1 e^x + c_2 xe^x$,
where $c_1$ and $c_2$ are arbitrary constants.

Final Trial solution. The derivatives of the RHS give independent functions
$e^x, xe^x, x^2 e^x$. An initial trial solution is $y = (d_1 + d_2 x + d_3 x^2)e^x$. There are
duplications of $y_h$ terms in $y$. The fixup rule implies $y$ should be multiplied
twice by $x$ to obtain the final trial solution $y = x^2(d_1 + d_2 x + d_3 x^2)e^x$.

Evaluate. Substitute the final trial solution into $y'' - 2y' + y = (1 + x - x^2)e^x$,
in order to find the undetermined coefficients $d_1, d_2, d_3$. To present the details,
let \( q(x) = x^2(d_1 + d_2 x + d_3 x^2) \), then \( y = q(x)e^x \) implies
\[
\begin{align*}
\text{LHS} &= y'' - 2y' + y \\
&= [q(x)e^x]''' - 2[q(x)e^x]' + q(x)e^x \\
&= q(x)e^x + 2q'(x)e^x + q''(x)e^x - 2q'(x)e^x - 2q(x)e^x + q(x)e^x \\
&= q''(x)e^x \\
&= [2d_1 + 6d_2 x + 12d_3 x^2]e^x.
\end{align*}
\]
Because \( \text{LHS} = \text{RHS} = (1+x-x^2)e^x \), then \( e^x \) cancels and \( 2d_1 + 6d_2 x + 12d_3 x^2 = 1 + x - x^2 \). Matching powers of \( x \) gives \( 2d_1 = 1, \ 6d_2 = 1, \ 12d_3 = -1 \). Then \( y = x^2(1/2 + x/6 - x^2/12)e^x \).

**Exercises 4.4**

**Polynomial Solutions.** Determine a polynomial solution \( y_p \) for the given differential equation.

1. \( y'' = x \)
2. \( y'' = x - 1 \)
3. \( y'' = x^2 - x \)
4. \( y'' = x^2 + x - 1 \)
5. \( y'' - y' = 1 \)
6. \( y'' - 5y' = 10 \)
7. \( y'' - y' = x \)
8. \( y'' - y' = x - 1 \)
9. \( y'' - y' + y = 1 \)
10. \( y'' - y' + y = -2 \)
11. \( y'' + y = 1 - x \)
12. \( y'' + y = 2 + x \)
13. \( y'' - y = x^2 \)
14. \( y'' - y = x^3 \)

**Polynomial-Exponential Solutions.** Determine a solution \( y_p \) for the given differential equation.

15. \( y'' + y = e^x \)
16. \( y'' + y = e^{-x} \)
17. \( y'' = e^{2x} \)
18. \( y'' = e^{-2x} \)
19. \( y'' - y = (x + 1)e^{2x} \)
20. \( y'' - y = (x - 1)e^{-2x} \)
21. \( y'' - y' = (x + 3)e^{2x} \)
22. \( y'' - y' = (x - 2)e^{-2x} \)
23. \( y'' - 3y' + 2y = (x^2 + 3)e^{3x} \)
24. \( y'' - 3y' + 2y = (x^2 - 2)e^{-3x} \)

**Sine and Cosine Solutions.** Determine a solution \( y_p \) for the given differential equation.

25. \( y'' = \sin(x) \)
26. \( y'' = \cos(x) \)
27. \( y'' + y = \sin(x) \)
28. \( y'' + y = \cos(x) \)
29. \( y'' = (x + 1)\sin(x) \)
30. \( y'' = (x + 1)\cos(x) \)
31. \( y'' - y = (x + 1)e^x \sin(2x) \)
32. \( y'' - y = (x + 1)e^x \cos(2x) \)
33. \( y'' - y' - y = (x^2 + x)e^x \sin(2x) \)
34. \( y'' - y' - y = (x^2 + x)e^x \cos(2x) \)

**Undetermined Coefficients Algorithm.** Determine a solution \( y_p \) for the given differential equation. These exercises require decomposition into easily-solved equations.
4.4 Undetermined Coefficients

35. \( y'' = x + \sin(x) \)

36. \( y'' = 1 + x + \cos(x) \)

37. \( y'' + y = x + \sin(x) \)

38. \( y'' + y = 1 + x + \cos(x) \)

39. \( y'' + y = \sin(x) + \cos(x) \)

40. \( y'' + y = \sin(x) - \cos(x) \)

41. \( y'' = x + xe^x + \sin(x) \)

42. \( y'' = x - xe^x + \cos(x) \)

43. \( y'' - y = \sinh(x) + \cos^2(x) \)

44. \( y'' - y = \cosh(x) + \sin^2(x) \)

45. \( y'' + y' - y = x^2e^{-x} + xe^x \cos(2x) \)

46. \( y'' + y' - y = x^2e^{-x} + xe^x \sin(2x) \)

Additional Proofs. The exercises below fill in details in the text.

47. (Superposition) Let \( Ly \) denote \( ay'' + by' + cy \). Show that solutions of \( Lu = f(x) \) and \( Lv = g(x) \) add to give \( y = u + v \) as a solution of \( Ly = f(x) + g(x) \).

48. (Easily Solved Equations) Let \( Ly \) denote \( ay'' + by' + cy \). Let \( Ly_k = f_k(x) \) for \( k = 1, \ldots, n \) and define \( y = y_1 + \cdots + y_n \), \( f = f_1 + \cdots + f_n \). Show that \( Ly = f(x) \).