Applied Differential Equations 2250-1 and 2250-3
Midterm Exam 4, Due classtime 27-Nov-2002

Instructions. The four take-home problems below are to be submitted by Wednesday, November 27. Answer checks are expected. If maple assist is used, then please attach the maple output.

The in-class portion of the exam (December 2) is 15 minutes, one problem, of a type similar to one of the last two problems. Calculators, hand-written or computer-generated notes are allowed, including xerox copies of tables or classroom xerox notes. Books are not allowed.

1. (Eigenanalysis) Find the 3×3 matrix \( P \) which under the change of variables \( x = PY \) converts the system \( x' = Ax \) into \( Y' = DY \), where

\[
A = \begin{pmatrix}
-1 & -7 & -3 \\
0 & 3 & 0 \\
0 & -1 & 2 \\
\end{pmatrix}, \quad D = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 3 \\
\end{pmatrix}.
\]

Represent the general solution of \( x' = Ax \) as a matrix product, by solving \( Y' = DY \) and then back-substituting the answer into the relation \( x = PY \).

2. (Coupled spring-mass system) The system

\[
\begin{align*}
x_1'' &= -k_1 x_1 + k_2 (x_2 - x_1), \\
x_2'' &= -k_2 (x_2 - x_1) + k_3 (x_3 - x_2), \\
x_3'' &= -k_3 (x_3 - x_2) - k_4 x_3
\end{align*}
\]

represents three masses \( m_1, m_2, m_3 \) coupled by springs of Hooke’s constant \( k_1, k_2, k_3, k_4 \) as in Figure 7.4.1, Edwards-Penney. Let \( m_1 = m_2 = m_3 = 1, k_1 = k_2 = k_3 = k_4 = 1 \). Find the natural frequencies \( \omega_1, \omega_2, \omega_3 \) of oscillation of system (1). Do Not Solve for \( x_1, x_2, x_3 \! \)!

3. (Laplace transform) Solve \( x'' + x = \sin 2t, x(0) = 0, x'(0) = 0 \) by two methods: (1) Undetermined coefficients and (2) Laplace transform. Show all steps, thus verifying the answer \( x = (2 \sin t - \sin 2t)/3 \).

4. (Laplace inverse transform) Show the partial fraction steps involved in solving for \( f(t) \) in the Laplace equation

\[
\mathcal{L}(f(t)) = \frac{2s}{(s-1)(s-2)(s^2+1)}.
\]

Kindly flag the step where Lerch’s theorem is applied to give the answer \( f(t) = -e^t + \frac{4}{5} e^{2t} + \frac{1}{5} \cos(t) - \frac{2}{5} \sin(t) \).