1. (ref)
   (a) Determine $b$ such that the system has infinitely many solutions:
   
   \[
   \begin{align*}
   x + 2y + z &= b \\
   3x + y + 2z &= 2b \\
   4x + 3y + 3z &= 1 + b
   \end{align*}
   \]

   Answer: $1 - 2b = 0$

   (b) Determine $a, b$ such that the system has infinitely many solutions:
   
   \[
   \begin{align*}
   x + 2y + z &= a \\
   5x + y + 2z &= 3a \\
   6x + 3y + bz &= 1 + a
   \end{align*}
   \]

   Answer: $-3 + b = 0$ and $3a - 1 = 0$

2. (vector spaces)
   (a) Give two examples of vector spaces of functions, one of dimension two and one of dimension three.
   (b) Let $V$ be the vector space of all continuous functions on the real line and let $S$ be the subset of $V$ given by all solutions of the differential equation $y' = -2y$. Prove that $S$ is a subspace of $V$.
   (c) Find a basis for the subspace of $\mathbb{R}^3$ given by the system of equations
   
   \[
   \begin{align*}
   x + 2y - z &= 0, \\
   x + y - 2z &= 0, \\
   y + z &= 0,
   \end{align*}
   \]

   Answer (a): $V = \{c_1 + c_2 t\}$ and $W = \{c_1 + c_2 t + c_3 t^2\}$ are vector spaces of polynomials with $\dim(V) = 2$ and $\dim(W) = 3$.

   Solution (b): All functions $y$ in $S$ look like $y = c_1 e^{-2t}$. Adding two such functions gives a function in $S$ and multiplying such a function by a scalar gives a function in $S$. Then $S$ is closed under addition and scalar multiplication. Therefore, $S$ is a subspace of $V$.

   Answer (c): $x = 3t_1$, $y = -t_1$, $z = t_1$. A basis is $\partial_{t_1} \mathbf{x} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$.

3. (independence)
   (a) Let $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$. State and apply a test that shows $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are dependent.

   (b) Extract from the list below a largest set of independent vectors.

   \[
   \begin{align*}
   \mathbf{a} &= \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}, \\
   \mathbf{b} &= \begin{pmatrix} 2 \\ -2 \\ 0 \\ -2 \end{pmatrix}, \\
   \mathbf{c} &= \begin{pmatrix} 3 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \\
   \mathbf{d} &= \begin{pmatrix} 0 \\ 2 \\ 0 \\ 4 \end{pmatrix}, \\
   \mathbf{e} &= \begin{pmatrix} 1 \\ 1 \\ 0 \\ 3 \end{pmatrix}.
   \end{align*}
   \]

   Solution (a): Let $A = \text{aug}(\mathbf{u}, \mathbf{v}, \mathbf{w})$. The test is
rref$(A)$ has three leading ones if and only if $u, v, w$ are independent.

Computing rref$(A)$ shows it has a row of zeros, so the vectors are dependent.
Solution (b): Let $A = \text{aug}(a, b, c, d, e)$. Find rref$(A)$. Then leading ones are in columns 1 and 3, giving corresponding columns $a, c$ as the pivot columns of $A$. Answer: \{a, c\} is a largest independent subset.

4. (determinants and elementary matrices)
(a) Assume given $3 \times 3$ matrices $A, B$. Suppose $B = E_1E_2A$ and $E_1, E_2$ are elementary matrices representing swap rules. Explain precisely why $\det(B) = \det(A)$.
(b) Let $A$ and $B$ be two $7 \times 7$ matrices such that $AB$ contains two duplicate rows. Explain precisely why either $\det(A)$ or $\det(B)$ is zero.
Solution (a): $\det(B) = \det(E_1)\det(E_2)\det(A)$ by the product rule for determinants. Each swap rule has determinant $-1$. So $\det(B) = (-1)(-1)\det(A) = \det(A)$.
Solution (b): $\det(AB) = 0$ because the determinant has two duplicate rows. Then $\det(A)\det(B) = 0$ by the product theorem for determinants. Hence either $\det(A) = 0$ or $\det(B) = 0$.

5. (inverses and Cramer’s rule)
(a) Determine all values of $x$ for which $A^{-1}$ exists: $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & -3 \\ 0 & x & 1 \end{pmatrix}$.
(b) Solve for $y$ in $Au = b$ by Cramer’s rule: $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 2 \\ 2 & -2 & 1 \end{pmatrix}$, $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.
Answer (a): $\det(A) \neq 0$ which is $x \neq 4/3$.
Answer (b): $\Delta = 6$, $x = 0$, $y = 1/2$, $z = 0$. The answer for $y$ only is obtained as a quotient of two determinants, so it takes less time than finding all three values.