Applied Differential Equations 2250-1 and 2250-2
Midterm Exam 2, Fall 2003, Version E-K
Due Wed 15 Oct (1,2) and Fri 17 Oct (3,4)
Inclass Exam Date: Monday, 20 October, 2003

Instructions. Choose the exam version based upon your last name, e.g., John Fox chooses exam version A-D, because F of Fox is between E and K.

The four problems below are take-home, due on the dates above at class time. Answer checks are expected. If maple assist is used, then please attach the maple output. The remaining 20% of the exam is in class, the last 15 minutes of the hour, one problem, of a type similar to # 3 or 4 below. No books, notes, calculators, computers or outside materials allowed.

1. (Periodic harvesting) The population equation \( y' = 2y(7 - y) - 11 \sin(2\pi t/5) \) appears to have a steady-state periodic solution that oscillates about \( y = 7 \). (a) Apply ideas from the example below to make a computer graphic with 6 solution curves that oscillate about \( y = 7 \). Submit the plot and the maple code. (b) Find by computer experiment a threshold population size \( y_1 \) so that \( y(0) < y_1 \) implies \( y(t) = 0 \) (population dies out) for some later time \( t \), while \( y(0) > y_1 \) implies \( y(t) > 0 \) forever and the solution \( y(t) \) oscillates about \( y = 7 \). See Figure 2.5.12, page 128.

# Example. See Figure 12, section 2.5
with(DEtools):
de:=diff(y(t),t)=y(t)*(2-y(t))-4*cos(4*Pi*t):
ic:=[y(0)=1.7], [y(0)=2], [y(0)=2.4], [y(0)=2.8]:
DEplot(de,y(t),t=0..4,y=1..3,[ic],stepsize=0.05);

2. (Jules Verne Problem) Assume a model

\[
\frac{d^2r}{dt^2} = -\frac{Gm_1}{(R_1 + r)^2} + \frac{Gm_2}{(R_2 - R_1 - r)^2}, \quad r(0) = 0, \quad r'(0) = v_0,
\]

where \( R_2 \) is the mean center-to-center distance from the earth to the moon and \( R_1 \) is the mean radius of the earth. The mass \( m_1 \) of the earth and \( m_2 \) of the moon appear, plus the universal gravitation constant \( G \). All units are MKS.

(a) Explain why this model takes into account the gravitational attraction of both the moon and the earth.

(b) Calculate the distance \( r^* \) at which the projectile has net acceleration zero. Give a symbolic answer and also a numerical answer \( \approx 3.39 \times 10^8 \) meters.
(c) Conduct a numerical experiment to find the flight time to the moon, when the launch velocity $r'(0)$ is 56 m/s faster than the minimal launch velocity $v_0 = \sqrt{2F(0) - 2F(r^*)}$, $F(r) = \frac{Gm_1}{R_1 + r} + \frac{Gm_2}{R_2 - R_1 - r}$. Use the sample maple code below to do the experiment.

```
# Group 1
G:=6.6726e-11: m1:=5.975e24: m2:=7.36e22:
R1:=6.378e6: R2:=3.84e8: v0:=1000: T:=210:
dev:=diff(r(t),t,t)=-G*m1/(r(t)+R1)^2+G*m2/(R2-R1-r(t))**2:
ic:=r(0)=0,D(r)(0)=v0:
p:=dsolve({de,ic},r(t),type=numeric,method=lsode):
Y:=t->rhs(p(t)[2]):
plot('Y(t)',t=0..T);
# Plot done. Change v0, T and re-execute group 1.
```

3. (Gaussian algorithm) Solve for $x$, $y$, $z$ in the $3 \times 3$ linear system

\[
\begin{align*}
2x &+ 2(a+b)y &+ cz &= b \\
-x &+ (a+b)y &+ cz &= b \\
x &+ 3(a+b)y &+ 2cz &= 2b
\end{align*}
\]

using the Gaussian algorithm, for all constant values of $a$, $b$, $c$. Include all algorithm details and an answer check for each of the three separate cases. Sanity check: $a + b \neq 0$ is one case, with parametric solution $x = -b/4 + ct_1/4$, $y = 3b/(4a + 4b) - 3ct_1/(4a + 4b)$, $z = t_1$. The case $a + b = 0$ has subcases $c \neq 0$ and $c = 0$, for one of which you will report no solution.

4. (Inverse matrix) Determine by rref methods the inverse matrix of

\[
A = \begin{pmatrix}
3 & b & 0 \\
a & 0 & b \\
0 & 1 & -1
\end{pmatrix}.
\]

Please state conditions on $a$, $b$ for when the inverse exists. Show all hand details. Prove that in the absence of your condition, no inverse exists. Include an answer check, preferably done in maple.