The rate of change of the indoor temperature is proportional to the difference between the ambient and indoor temperatures.

This implies that \( du/dt \) is proportional to \( A - u \). Let \( A(t) \) be the ambient outside temperature and let \( k \) denote a constant. A model for the indoor temperature \( u(t) \) is given by \( du/dt = k(A - u) \), \( u(0) = u_0 \), which can be rearranged as

\[
u'(t) + ku(t) = kA(t), \quad u(0) = u_0.
\]

The number \( k \) is called the insulation constant.

**Ambient Temperature Model.** Let \( M = 54, m = 22, \omega_0 = \pi/12 \). The ambient temperature formula

\[
A(t) = \frac{1}{2}(M + m) - \frac{1}{2}(M - m) \cos \omega_0(t - 3)
\]

satisfies \( M = \max A(t) = A(15) \), \( m = \min A(t) = A(3) \) and \( A(t) \) is \( t \)-periodic of period 24 hours.

For use in maple, a function of two variables

\[
AA(t, \omega) = \frac{1}{2}(M + m) - \frac{1}{2}(M - m) \cos \omega(t - 3)
\]

is used to maintain the variable name \( \omega \) in displays.

**Indoor Temperature** \( u(t) \). The integrating factor method for linear equations applies to find the general solution by these steps:

- \( u' + ku = kA(t) \) Copy the differential equation. The integrating factor is \( e^{kt} \).
- \( \frac{(e^{kt}u)'}{e^{kt}} = kA(t) \) Replace the left side.
- \( (e^{kt}u)' = kA(t)e^{kt} \) Clear fractions.
- \( e^{kt}u = u_0 + \int_0^t kA(x)e^{kx} \, dx \) Integrate both sides with respect to \( t \). Apply the Fundamental Theorem of Calculus. Use \( u(0) = u_0 \).
- \( u = u_0 e^{-kt} + e^{-kt} \int_0^t kA(x)e^{kx} \, dx \) Divide to isolate \( u \).
- \( u = u_0 e^{-kt} + k \int_0^t e^{k(x-t)} A(x) \, dx \) Exponential rule: \( e^a e^b = e^{a+b} \).

Let \( u_h(t) = u_0 e^{-kt} \), a solution of the homogeneous differential equation \( u' + ku = 0 \). Let \( u_p(t) = k \int_0^t e^{k(x-t)} A(x) \, dx \), a particular solution of the nonhomogeneous differential equation \( u' + ku = kA(t) \). Then the indoor temperature \( u = u_h + u_p \) depends on the time \( t \), the initial temperature \( u_0 \), the insulation constant \( k \) and the frequency \( \omega \) (which is fixed at \( \pi/12 \)). Write \( u = u(t, u_0, k, \omega) \) to emphasize the dependence. In maple, advantages exist for adding the variable name \( \omega \), which is later set to value \( \omega_0 = \pi/12 \). Write \( u \) as \( U(t, u_0, k, \text{omega}) \) for use in maple.
Steady-state solution. The steady-state solution $u_{ss}$ is obtained from the general formula $u = u_h + u_p$ by dropping all terms containing a negative exponential. It depends on $t$, $k$, and $\omega$ but it is independent of $u_0$.

Problem 1.3. (Solution formulas for $u_p$ and $u$)
Derive by hand, using integral tables, an explicit symbolic formula for $u_p(t)$. Display a final formula for $u = u_h + u_p$ which depends only on $t$, $u_0$, $k$ and $\omega$. Check your hand answer for $u$ in maple. The only maple assist in this problem is the answer check.

Problem 1.4. (Steady-state Periodic Solution)
Derive by hand a formula for the steady-state periodic solution $u_{ss}$ of $u' + ku = kA(t)$. The only maple assist in this problem is an answer check.

Problem 1.5. (Indoor-Outdoor Variation)
Compare in a maple graphic the indoor ($u(t)$) and outdoor ($A(t)$) temperature oscillations over a 48-hour period assuming $k = 0.3$, $u_0 = 69$, $\omega = \pi/12$. Compute the indoor temperature variation from this plot. Find the phase delay using a second plot of steady-state ($u_{ss}(t)$) and outdoor ($A(t)$) temperatures.

Problem 1.6. (Freezing Pipes)
Assume $\omega = \pi/12$ and the insulation constant $k$ ranges from 0.1 to 0.5. Suppose the inside temperature is 69 degrees at midnight when the furnace is turned off. Report approximate ranges of hours during the first 72 hours, for which the indoor temperature is at or below 31 degrees. Justify your logic used to find the ranges, in a short paragraph. Illustrate with a computer graphic.

Problem notes.

Notes on 1.3: The integration problem to be solved by hand using the book’s integral tables is

$$u_p(t) = ke^{-kt} \int_0^t e^{kx}(38 - 16 \cos \omega(x - 3)) dx.$$

A change of variable $u = x - 3$ allows use of integral table entry $\int e^{au} \cos bu \, du$. The symbol $\omega$ will be set to $\pi/12$, but for simplicity, use symbol $\omega$ throughout.

The formulae for $u_h$ and $u_p$ are used again in 1.4 in order to derive the steady-state solution. Your answer must contain symbols $t$, $u_0$, $k$, $\omega$.

The answer check in maple is organized as follows. The complications of setting $\omega = \pi/12$ are avoided here by leaving $\omega$ as a symbol, since it does not affect the answer check.

```maple
# Test LHS=RHS for u'+ku=kA.
t:=t: u0:='u0': omega:='omega': k:='k': myANS:=your hand-derived formula for u=uh+up:
LHS:=diff(myANS,t)+k*myANS:
RHS:=k*(38-16*cos(omega*(t-3))):
simplify(expand(LHS-RHS));
```

A successful test of $LHS = RHS$ produces answer zero, or an expression that reduces to zero.

The algebra system maple can verify table use, for example, `int(exp(k*x)*cos(w*(x-3)),x=0..t)` will check some of the hand integration details.

Notes on 1.4: The steady-state solution is derived from the hand-generated symbolic solution $u = u_h + u_p$ in 1.3 by dropping all terms that contain $e^{-kt}$. The answer, where $\omega_0 = \pi/12$:

$$u_{ss} = 38 - \frac{16k}{k^2 + \omega_0^2}(k \cos \omega_0(t - 3) + \omega_0 \sin \omega_0(t - 3)).$$

To check your answer, use maple as in 1.3.

To get maple to report the above formula, it is essential to evaluate everything with $\omega$ as a symbol, to wit, use in maple the statement `omega:='omega'` Beware of writing `omega:=Pi/12` which defines $\omega$ to be a constant, unless you undo the effect immediately afterward.

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Notes on 1.5: The outside temperature variation $A(t)$ ($AA(t,\omega)$ in maple) and the solution $u = U(t,u_0,k,\omega)$ obtained in 1.3 can be programmed in maple as follows:

```maple
with(Student[Calculus1]);
t:=t;u0:=u0;k:=k;omega:=omega;
AA:=(t,omega)->38-16*cos(omega*(t-3));
U:=(t,u0,k,omega)-> your answer of u=uh+up from 1.3:

The two plots are placed onto one graphic by this maple command:

```maple
plot({U(t,69,0.3,Pi/12),AA(t,Pi/12)},t=0..48);
``` 

Missing curves? Probably, the missing curves are defined to contain an unevaluated variable name, like $\pi$ instead of $\text{Pi}$ While $\text{Pi}$ is the constant 3.14159, $\pi$ is a variable name: case is significant in maple. Curves vanish on the printer? Add `color=black` to the plot command.

Click with the left mouse button on the high and low spots in the graphic. Somewhere on the maple worksheet the coordinates of the click are displayed, and this is enough to find a good approximation to the max and min values.

The indoor temperature variation is just the maximum minus the minimum, as computed from the first plot. Beware: $u$ has a maximum at $t = 0$.

The phase shift is found from a second plot, which uses the steady-state solution $u_{\text{ss}}(t,k,\omega)$ obtained in 1.4:

```maple
plot({uss(t,0.3,Pi/12),AA(t,Pi/12)},t=0..48);
``` 

The shift is computed as $|T_2 - T_1|$, where $A(T_1) = \max A(t)$ and $u_{\text{ss}}(T_2) = \max u_{\text{ss}}(t)$. Look at the graphic to find sane answers for $T_1$ and $T_2$. See the textbook for a more complete discussion of the ideas. Beware: $T_1$ and $T_2$ are less than 20.

Notes on 1.6: A computer algebra assist for this problem can be found in maple’s function `implicitplot`. This function can plot the equation $u(t,69,k,\pi/12) = 31$ over the domain $0 \leq t \leq 72, 0.2 \leq k \leq 0.5$. From this plot, and the 3D-plot $z = u(x,69,y,\pi/12)$, the question is easily answered.

```maple
with(plots):
t:=t;u0:=u0;k:=k;omega:=omega;
U:=(t,u0,k,omega)-> your answer of uh+up from 1.3:
implicitplot(U(t,69,k,Pi/12)=31,t=0..72,k=0.2..0.5);
plot3d({U(t,69,k,Pi/12),31},t=0..72,k=0.2..0.5);
``` 

Zoom in on the implicit plot by using a smaller time domain, suggested by the larger plot. The relation between the implicit plot and the 3D plot is seen by slicing the 3D plot at height $z = 31$ to obtain a “bread slice,” depicted in the implicit plot as that “bread slice” projected into the $xy$-plane.

Physically, inside temperature 31 degrees is reached a few hours after the outside temperature drops below 31 degrees. During 72 hours, there are three such inside temperature drops (see the 3D-plot, where $z=$temperature). For $k < 0.30$, inside temperature 31F is not reached in the first 23 hours. Please report an answer like $9-12, 37-39, 56-59$ hours plus an explanation of the logic applied to obtain this answer from the graphics. It is easiest to give two reports, one for $0.31 < k < 0.5$ and one for $0 < k < 0.31$. 
