Midterm 1

INSTRUCTIONS:

1. Calculators are NOT allowed.

2. REMEMBER: Don’t spend too much time on problems which are worth very few points.

3. Don’t get stuck on a problem, KEEP MOVING ON and in the end come back to the problems you could not figure out on first try.

4. Show proper work to get full points. If your answer is wrong you still have a chance of getting partial credit for the work.

5. TRUE/FALSE: Remember if a statement is false you can give an example not satisfying the statement as a proof but if the statement is true an example is not sufficient.

Maximum Points = 50 points
Number of Pages : 7

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BEST OF LUCK!!
1. [10 points] State True/False. Justify your answer to get full points:

(a) If \( \vec{u} \) is a scalar multiple of \( \vec{v} \) then \( \vec{u} \times \vec{v} = 0 \).
   **TRUE**
   \[ \vec{u} = c \vec{v} \text{ for some } c \in \mathbb{R}, c \neq 0 \]
   i.e. \( \vec{u} \) is parallel to \( \vec{v} \) therefore \( \vec{u} \times \vec{v} = \vec{0} \).

(b) If \( \vec{v}(t), \vec{v}(t) \) is a constant, then \( \vec{v}(t), (\vec{v}(t))' = 0 \).
   **TRUE**
   \[ \frac{d}{dt} (\vec{v}(t), \vec{v}(t)) = 0 \]
   \[ \Rightarrow \vec{v}(t), \vec{v}(t) = 0 \]

(c) The graph of the equation \( \phi = 0 \) is the z-axis. (\( \phi \) is a spherical coordinate)
   **FALSE**
   The graph of \( \phi = 0 \) is positive z-axis

(d) Let \( \vec{u} = 2\hat{i} + 3\hat{j} \) and \( \vec{v} = \hat{i} \). Then \( \text{Proj}_{\vec{v}} \vec{u} = \hat{i} \).
   **FALSE**
   Projection of \( \vec{u} \) on \( \vec{v} \) is \( \hat{i} + 2 \hat{j} \)

(e) The planes \( x + y + z = 1 \) and \( 2x + 2y + 2z = 1 \) intersect in a line.
   **FALSE**
   normal for plane \( P_1 \) is \( \hat{i} + \hat{j} + \hat{k} \)
   normal for plane \( P_2 \) is \( 2(\hat{i} + \hat{j} + \hat{k}) \)
   since the two normal vectors are parallel
   the two planes are also parallel
   so they do NOT intersect in a line.
2. [10 points] Find the equation of the plane containing the line

\[ x = 1 + 2t, \quad y = -1 + 3t, \quad z = 4 + t \]

and the point \((1, -1, 5)\).

A point on the line is \((1, -1, 4)\) (for \(t = 0\))

The vector joining \((1, -1, 4)\) and \((1, -1, 5)\) lies on the plane (or is parallel to the plane)

\[ \overrightarrow{\text{u}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \text{a normal to the plane} \]

Also, the vector \(2\hat{\imath} + 3\hat{j} + \hat{k}\) is parallel to the line

so, we have two vectors \(\overrightarrow{\text{u}}\) and \(\overrightarrow{\text{v}}\) parallel to the plane.

Then \(\overrightarrow{\text{u}} \times \overrightarrow{\text{v}}\) is normal to the plane.

\[
\begin{vmatrix}
\hat{\imath} & \hat{j} & \hat{k} \\
2 & 3 & 1 \\
0 & 0 & 1 \\
\end{vmatrix} = \hat{\imath}(3) - \hat{j}(2) + \hat{k}(0) = 3\hat{\imath} - 2\hat{j}
\]

Equation of plane

\[ 3(x-1) - 2(y+1) + 0(z-5) = 0 \]

\[ 3x - 2y - 3 - 2 = 0 \]

\[ 3x - 2y = 5 \quad \text{Ans.} \]
3. [10 points] Consider the curve $\vec{r}(t) = 2ti + t^2j + (1-t^2)k$.

(a) Find the equation of the tangent line at $t = 2$.
(b) Where does the tangent line at $t = 2$ intersect the $xy$-plane?
(c) (Extra Credit) : Show that this curve lies on a plane.

\[ \vec{r}(t) = 2t\hat{i} + t^2\hat{j} + (1-t^2)\hat{k} \quad \vec{r}'(2) = 4\hat{i} + 4\hat{j} - 2\hat{k} \]
\[ \vec{r}'(t) = 2\hat{i} + 2t\hat{j} - 2t\hat{k} \quad \vec{r}'(2) = 2\hat{i} + 4\hat{j} - 4\hat{k} \]

**Equation of line:**

\[
\begin{align*}
\frac{x}{2} &= \frac{y-4}{4} = \frac{z+3}{-4} \\
\end{align*}
\]

\[
\begin{align*}
(x, y, z) &= (\frac{5}{2}, 1, 0) \\
\end{align*}
\]

(b) $z$ coordinate is zero on $xy$ plane

So $0 = -3-4t$ \[ \Rightarrow t = -\frac{3}{4} \]

\[ x = y + \frac{3}{2} \]
\[ y = y + y = \frac{3}{4} \]

(c) To show that the curve lies on a plane, we will show that the normal vector does not depend on $t$.

\[ \vec{u} \times \vec{v} = \left| \begin{array}{ccc}
\hat{i} & \hat{j} & \hat{k} \\
2t & t^2 & -t^2 \\
2t & t^2 & -t^2 \\
\end{array} \right| = 2t\hat{i} + t^2\hat{j} - t^2\hat{k} \]

\[ \vec{u} \times \vec{v} = \left| \begin{array}{ccc}
\hat{i} & \hat{j} & \hat{k} \\
2t & t^2 & -t^2 \\
2t & t^2 & -t^2 \\
\end{array} \right| = 2t\hat{i} + t^2\hat{j} - t^2\hat{k} \]

The normal vector is just a scalar multiple of $\hat{k} - \hat{j}$.

Since $\hat{k} - \hat{j}$ is independent of $t$ at $t_1$, the curve lives in a plane.
4. [10 points]

(a) Graph the surface \( x^2 - 2y^2 + 3z^2 = 1 \).

Taking \( y \) to slies we get bigger and bigger ellipses as we move away from the origin along \( y \)-axis.

(b) Draw level sets for the function \( f(x, y, z) = e^{x^2+y^2+z^2} \) for \( f(x, y, z) = -1, 0, 1, 2 \).

1. \( f(x, y, z) = -1 \)
   \[ e^{x^2+y^2+z^2} = -1 \]
   There are no possible values of \((x, y, z)\) satisfying this.

2. \( f(x, y, z) = 0 \)
   \[ e^{x^2+y^2+z^2} = 0 \]
   Again there are no values of \((x, y, z)\) satisfying this.

3. \( f(x, y, z) = 1 \)
   \[ e^{x^2+y^2+z^2} = 1 \]
   \( x^2+y^2+z^2 = \ln 1 = 0 \)
   Only \((0, 0, 0)\) satisfies this.

4. \( f(x, y, z) = 2 \)
   \[ e^{x^2+y^2+z^2} = 2 \]
   \( x^2+y^2+z^2 = \ln 2 \)
   This is a sphere of radius \( \sqrt{\ln 2} \) based at origin.
5. [5 points] Find the slope of the tangent line to the curve of intersection of the surface $z = x^2y$ and the plane $y = 3$ at the point $(1, 3, 3)$.

$$\frac{\partial z}{\partial x} = 2xy$$

Since $y$ is constant, we want $\frac{\partial z}{\partial x}$

$$\frac{\partial z}{\partial x} \bigg|_{(1,2,3)} = 2(1)(3) = 6$$

6. [5 points] Show that the following limit does not exist.

$$\lim_{(x,y) \to (0,0)} \frac{\sin(x^2 + 2y^2)}{x^2 + y^2}$$

[Hint: Choose two different lines through the origin along which limits are different.]

$$\lim_{(x,y) \to (0,0), \text{along } y=0} \frac{2 \sin(2y^2) \cdot x^2}{(x^2 + y^2)} = \lim_{y \to 0} \frac{\sin(2y^2)}{y^2} = 2$$

$$\lim_{(x,y) \to (0,0), \text{along } x=0} \frac{\sin(x^2 + 2y^2)}{x^2 + y^2} = \lim_{x \to 0} \frac{\sin(x^2)}{x^2} = 1$$

Since the two limits are different,

$$\lim_{(x,y) \to (0,0)} \frac{\sin(x^2 + 2y^2)}{x^2 + y^2}$$

does NOT exist. \(\square\).
1. Length of a parametrized curve $\mathbf{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ from $t = a$ to $t = b$ is given by

$$\int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} dt$$

2. Distance $d$ between a point $(x_0, y_0, z_0)$ and a plane $Ax + By + Cz = D$ is given by

$$d = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}$$

3. Volume of parallelepiped spanned by the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is given by

$$|\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}|$$