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Chapter 1

Abstract Measure Spaces

1.1 Basic Definitions

The discussion will be based on Stein’s *Real Analysis*.

**Definition 1.** A measure space consists of a set $X$ equipped with:

1. A non-empty collection $\mathcal{M}$ of subsets of $X$ closed under complements and countable unions and intersections (a $\sigma$-algebra), which are the “measurable” sets.

2. A measure $\mathcal{M} \xrightarrow{\mu} [0, \infty]$ with the property that if $E_1, E_2, \ldots$ is a countable family of disjoint sets in $\mathcal{M}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$$

Now we state an assumption on a measure space that allows us to develop integration as in the case of $\mathbb{R}^d$.

**Definition 2.** The measure space $(X, \mathcal{M}, \mu)$ is $\sigma$-finite if $X$ can be written as the union of countably many measurable sets with finite measure. (To see that $\mathbb{R}^d$ is $\sigma$-finite, consider balls centered at the origin with radii going to infinity.)

From now on, we assume $(X, \mathcal{M}, \mu)$ is a $\sigma$-finite measure space.

1.2 Measurable Functions

**Definition 3.** A function $X \xrightarrow{f} [-\infty, \infty]$ is measurable if $f^{-1}([-\infty, a])$ is measurable for all $a \in \mathbb{R}$.

**Proposition 1.** Some properties of measurable functions.
• If \( f, g \) are measurable and finite-valued a.e., then \( f + g \) and \( fg \) are measurable.

• If \( \{f_n\}_{n=1}^{\infty} \) is a sequence of measurable functions, then

\[
\lim_{n \to \infty} f_n(x), \quad \sup_{n} f_n(x), \quad \inf_{n} f_n(x), \quad \limsup_{n \to \infty} f_n(x), \quad \liminf_{n \to \infty} f_n(x)
\]

are measurable (lim only if it exists).

• If \( f \) is measurable and \( f(x) = g(x) \) for a.e. \( x \), then \( g \) is measurable.

Definition 4. A \textit{simple function} on \( X \) is of the form

\[
\sum_{k=1}^{N} a_k \chi_{E_k},
\]

where \( E_k \) are measurable sets of finite measure and \( a_k \) are real numbers.

Proposition 2. If \( f \) is a non-negative measurable function, then there exists a sequence of simple functions \( \{\varphi_k\}_{k=1}^{\infty} \) that satisfies

\[
\varphi_k(x) \leq \varphi_{k+1}(x) \quad \text{and} \quad \lim_{k \to \infty} \varphi_k(x) = f(x) \quad \text{for all} \quad x.
\]

In general, if \( f \) is only measurable, then we conclude that

\[
|\varphi_k(x)| \leq |\varphi_{k+1}(x)| \quad \text{and} \quad \lim_{k \to \infty} \varphi_k(x) = f(x) \quad \text{for all} \quad x.
\]

Proposition 3 (Egorov). Suppose \( \{f_k\}_{k=1}^{\infty} \) is a sequence of measurable functions defined on a measurable set \( E \subseteq X \) with \( \mu(E) < \infty \), and \( f_k \to f \) a.e. Then for each \( \epsilon > 0 \), there is a set \( A_\epsilon \subseteq E \), such that \( \mu(E - A_\epsilon) \leq \epsilon \) and \( f_k \to f \) uniformly on \( A_\epsilon \).

1.3 Integration

We define the \textit{Lebesgue integral} \( \int_X f(x) \, d\mu(x) \), often written \( \int f \), of a general measurable function \( f \) by a four part process:

Definition 5. Let \( f \) be a measurable function.

(i) If \( f(x) = \sum_{k=1}^{N} a_k \chi_{E_k}(x) \) is a simple function, then

\[
\int f := \sum_{k=1}^{N} a_k \mu(E_k).
\]

(ii) If \( f \) is bounded and supported on a set of finite measure, then choose a sequence of simple functions \( \{\varphi_k\}_{k=1}^{\infty} \) such that \( |\varphi_k(x)| \leq |\varphi_{k+1}(x)| \) and \( \lim_{k \to \infty} \varphi_k(x) = f(x) \) for all \( x \). Then

\[
\int f := \lim_{n \to \infty} \int \varphi_n,
\]
(iii) If \( f \) is non-negative, then
\[
\int f := \sup_g \int g,
\]
where the supremum is taken over all measurable functions \( g \) such that \( 0 \leq g \leq f \) and \( g \) is bounded and supported on a set of finite measure.

(iv) For general \( f \), write \( f = f^+ - f^- \), a decomposition into non-negative measurable functions. Then
\[
\int f := \int f^+ - \int f^-.
\]

**Definition 6.** Let \( f \) be measurable. Then \( f \) is **Lebesgue integrable** if \( \int |f| < \infty \) in the sense of (iii) above.

**Proposition 4.** Key properties of the Lebesgue integral are:

- **Linearity.** If \( f, g \) integrable and \( a, b \in \mathbb{R} \), then
  \[
  \int (af + bg) = a \int f + b \int g.
  \]

- **Additivity.** If \( E, F \) are disjoint measurable subsets of \( X \), then
  \[
  \int_{E \cup F} f = \int_E f + \int_F f.
  \]

- **Monotonicity.** If \( f \leq g \), then
  \[
  \int f \leq \int g.
  \]

- **Triangle inequality.**
  \[
  \left| \int f \right| \leq \int |f|.
  \]

**Proof.** Prove these step by step, as in the definition of the integral, starting with the simple functions. The triangle inequality follows from monotonicity and linearity by considering \( f \leq |f| \) and \( -f \leq |f| \).

**Proposition 5.** Key limit theorems.

- **Fatou’s lemma.** If \( \{f_n\} \) is a sequence of non-negative measurable functions on \( X \), then
  \[
  \int \liminf_{n \to \infty} f_n \leq \liminf_{n \to \infty} \int f_n.
  \]
• **Monotone convergence.** If \( \{ f_n \} \) is a sequence of non-negative measurable functions with \( f_n \nearrow f \), then

\[
\lim_{n \to \infty} \int f_n = \int f.
\]

• **Dominated convergence.** If \( \{ f_n \} \) is a sequence of measurable functions with \( f_n \to f \) a.e., and such that \( |f_n| \leq g \) for some integrable \( g \), then

\[
\int |f_n - f| \to 0 \quad \text{as } n \to \infty,
\]

and consequently

\[
\int f_n \to \int f \quad \text{as } n \to \infty.
\]

**Proof.** Fatou’s lemma: Set \( f := \liminf_{n \to \infty} f_n \). Suppose \( 0 \leq g \leq f \), where \( g \) is bounded and supported on a set \( E \) of finite measure. Then \( g_n(x) := \min\{g(x), f_n(x)\} \) is measurable, supported on \( E \), and \( g_n(x) \to g(x) \), so that \( \int g_n \to \int g \) by the bounded convergence theorem (combine boundedness with Egorov to show \( \int |g_n - g| \to 0 \) as \( n \to \infty \)). Since \( g_n \leq f_n \), monotonicity gives

\[
\int g_n \leq \int f_n,
\]

whence

\[
\int g \leq \liminf_{n \to \infty} \int f_n.
\]

Since \( \int f = \sup g \int g \) for such \( g \), we get the result.

Monotone convergence: \( \int f_n \leq \int f \) by monotonicity, whence \( \limsup_{n \to \infty} \int f_n \leq \int f \). Conversely, Fatou’s lemma implies \( \int f \leq \liminf_{n \to \infty} \int f_n \). Combining these inequalities gives the result.

Dominated convergence: we will use the reverse Fatou’s lemma, which states that if \( \{ f_n \} \) is a sequence of measurable functions, such that \( f_n \leq g \) for some integrable function \( g \), then

\[
\int \limsup_{n \to \infty} f_n \geq \limsup_{n \to \infty} \int f_n.
\]

This is proved by applying Fatou’s lemma to the non-negative sequence \( \{ g - f_n \} \).

Since \( |f_n - f| < 2g \), the reverse Fatou’s lemma implies

\[
\limsup_{n \to \infty} \int |f_n - f| \leq \int \limsup_{n \to \infty} |f_n - f| = 0,
\]

whence we get the desired result using the linearity of the integral and the triangle inequality.

\[\square\]

### 1.4 Counterexamples

• A non-measurable subset of \( \mathbb{R} \) (Stein 24). For \( x, y \in [0, 1] \), consider the equivalence relation \( x \sim y \) whenever \( x - y \) is rational. Using the axiom
of choice, let \( \mathcal{N} \) be the set obtained by choosing one element from each equivalence class. Then if \( \{r_k\} \) is an enumeration of the rationals in \([-1, 1]\), we have

\[
[0, 1] \subseteq \bigcup_{k=1}^{\infty} (\mathcal{N} + r_k) \subseteq [-1, 2],
\]

but the measure of each \( \mathcal{N} + r_k \) is the same since they are translates of \( \mathcal{N} \). But \( 1 \leq \sum_{k=1}^{\infty} m(\mathcal{N} + r_k) \leq 3 \), a contradiction.

- A non-measurable function \( \mathbb{R} \xrightarrow{f} \mathbb{R} \) such that \( |f| \) is measurable. Let \( \mathcal{N} \subseteq \mathbb{R} \) be non-measurable, e.g. as above. Then

\[
f(x) = \begin{cases} 
1 & x \in \mathcal{N} \\
-1 & x \notin \mathcal{N}
\end{cases}
\]

is not measurable since \( f^{-1}((-\infty, 0)) = \mathcal{N}^c \) is not measurable, but \( |f| \equiv 1 \) is measurable.

- A sequence \( \{\varphi_n\} \) of non-negative bounded, measurable functions on \( \mathbb{R} \) that converge pointwise to an integrable function, where the \( \int \varphi_n \) do not converge to \( \int f \). Let

\[
\varphi_n(x) = \begin{cases} 
\frac{1}{2^n} & -n \leq x \leq n \\
0 & \text{otherwise}.
\end{cases}
\]

Then \( \varphi_n \to 0 \) pointwise, but \( \int \varphi_n = 1 \) for each \( n \). See also Stein 61.

- A Cauchy sequence \( \{f_n\} \) in the \( L^1 \) norm that does not converge pointwise a.e. Let \( f_1 = \chi_{[0,1]} \), \( f_2 = \chi_{[0,\frac{1}{2}]} \), \( f_3 = \chi_{[\frac{1}{2},1]} \), \( f_4 = \chi_{[0,\frac{1}{4}]} \), etc. (Stein 92, Ex. 12).

- A function \( f \) integrable but discontinuous at every \( x \), even when corrected on a set of measure 0. Set

\[
f(x) = \begin{cases} 
x^{-1/2} & \text{if } 0 < x < 1 \\
0 & \text{otherwise}.
\end{cases}
\]

Then if \( \{r_k\} \) is an enumeration of the rationals,

\[
F(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n)
\]

is integrable, so the series converges a.e., but the series is unbounded on every interval, and any function that agrees with \( F \) a.e. is unbounded in any interval (Stein 92-93, Ex. 15).
Chapter 2

Banach Spaces

2.1 Basic Definitions

Definition 7. A metric (distance measure) on a set $S$ is a function $S \times S \xrightarrow{d} \mathbb{R}$ such that
1. $d(x, y) \geq 0$;
2. $d(x, y) = 0 \iff x = y$;
3. $d(x, y) = d(y, x)$;
4. $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 8. A norm on a vector space $X$ is a function $X \xrightarrow{\|\cdot\|} \mathbb{R}$ such that
1. $\|x\| \geq 0$;
2. $\|x\| = 0 \iff x = 0$;
3. $\|\alpha x\| = |\alpha|\|x\|$, $\alpha \in \mathbb{F}$;
4. $\|x + y\| \leq \|x\| + \|y\|$.

One can use a norm to define a metric by setting
$$d(x, y) := \|x - y\|.$$ 

The metric triangle inequality holds since
$$d(x, z) = \|x - z\| = \|(x - y) + (y - z)\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z).$$

In general, a metric is not sufficient to define a norm.

Definition 9. A Banach space is a complete normed vector space. A Hilbert space is an inner-product space that is complete in the associated norm, hence also a Banach space. A separable Hilbert space has a countable spanning subset, hence has a (countable) orthonormal basis by Gram-Schmidt.
2.2 $L^p$ Spaces

**Proposition 6.** Let $1 \leq p < \infty$ and $(X, \mu)$ a measure space. The set of equivalence classes of measurable functions $X \xrightarrow{f} \mathbb{C}$ (that agree a.e.) such that

$$\|f\|_p := \left( \int_X |f|^p \right)^{1/p} < \infty$$

is a normed vector space $L^p(X, \mu)$.

**Proof.** First we check that $L^p$ is a vector space. It is clearly closed under (e.g. complex) scalar multiplication. For vector addition, note that $|f(x) + g(x)| \leq 2 \max\{|f(x)|, |g(x)|\}$, whence $|f(x) + g(x)|^p \leq 2^p(|f(x)|^p + |g(x)|^p)$.

Now we check the properties of the norm. Clearly $\|f\|_p \geq 0$ for all $f$, and $\|f\|_p = 0$ if and only if $f = 0$. Likewise, $\|\lambda f\|_p = |\lambda| \|f\|_p$ is clear by the linearity of the integral. The triangle inequality is true but difficult to prove – look up Minkowski’s inequality. \[\square\]

For $p = \infty$, consider the equivalence classes of measurable functions that are bounded except on a set of measure 0, with the norm

$$\|f\|_{\infty} := \inf\{C \geq 0 : |f(x)| \leq C \text{ a.e. } x\}.$$

Then

$$\|f\|_{\infty} = \lim_{p \to \infty} \|f\|_p$$

holds if $f$ is in $L^p(X, \mu)$ for some $p < \infty$. (If $f \in L^p$ and $f \in L^\infty$, then $f \in L^q$ for each $p \leq q \leq \infty$. For the set $E$ on which $|f(x)| \geq 1$ must have finite measure, so we can bound $f$ on this set by some $M$, and obtain $\int |f|^q \leq \int |f|^p + \mu(E) \cdot M$.)

The fact that $L^p(X, \mu)$ for $1 \leq p \leq \infty$ is complete (and hence a Banach space) is the **Riesz-Fischer theorem**. When $p < \infty$, $L^p$ is separable (for $X = \mathbb{R}^d$, the countable collection is $r\chi_R(x)$, where $r$ is a complex number with rational real and imaginary parts and $R$ is a rectangle with rational coordinates).

In the special case when $p = 2$, $L^2$ is in fact a Hilbert space under the inner product

$$(f, g) = \int f(x)g(x)dx.$$

A useful tool in the study of $L^p$ spaces is **Hölder’s inequality**, which states that if $f$ and $g$ are measurable and $1 \leq p, q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q,$$

namely

$$\int |fg| \leq \left( \int |f|^p \right)^{1/p} \left( \int |g|^q \right)^{1/q}.$$

In particular, if $f \in L^p$ and $g \in L^q$, then $fg \in L^1$. 

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2.3 \( \ell^p \) Spaces

Consider the vector space of sequences over \( \mathbb{R} \) or \( \mathbb{C} \). For \( 0 < p < \infty \), \( \ell^p \) is the subspace of all sequences \( \mathbf{x} = \{x_k\} \) such that

\[
\sum_{k \geq 1} |x_k|^p < \infty.
\]

If \( p \geq 1 \), then

\[
\| \mathbf{x} \|_p := \left( \sum_{k \geq 1} |x_k|^p \right)^{1/p}
\]

defines a norm on \( \ell^p \), and \( \ell^p \) is complete, hence a Banach space (it’s the \( L^p \) space on the integers with the counting measure). If \( 0 < p < 1 \), then \( \ell^p \) does not have a norm, but still has the metric

\[
d(\mathbf{x}, \mathbf{y}) = \sum_{k \geq 1} |x_k - y_k|^p.
\]

(To see that the natural attempt to define a norm fails to satisfy the triangle inequality, let \( p = \frac{1}{2} \) and set

\[
\mathbf{x} = (\frac{1}{2}, 0, 0, \ldots), \quad \mathbf{y} = (0, \frac{1}{2}, 0, \ldots).
\]

Then

\[
\| \mathbf{x} + \mathbf{y} \| = \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right)^2 = 2,
\]

while

\[
\| \mathbf{x} \| + \| \mathbf{y} \| = \frac{1}{2} + \frac{1}{2} = 1.
\]

If \( p = \infty \), then \( \ell^\infty \) is the space of all bounded sequences, which is a Banach space under the norm

\[
\| \mathbf{x} \|_\infty := \sup_k |x_k|.
\]

Note that

\[
\ell^p \subset \ell^q \subset \ell^\infty
\]

for \( 1 \leq p < q < \infty \) (since only finitely many elements of the sequence can have absolute value \( \geq 1 \)). Also, \( \ell^2 \) is the only \( \ell^p \) space that is a Hilbert space.
Chapter 3

Uniform Boundedness Principle

The discussion in this chapter, based on Chapter 8 of Munkres, is geared toward proving the Uniform Boundedness Principle.

**Definition 10.** A topological space $X$ is a Baire space if: Given any countable collection $\{A_n\}$ of closed sets of $X$ each of which has empty interior in $X$, their union $\bigcup A_n$ also has empty interior in $X$.

**Example 1.**
- $\mathbb{Q}$ in the standard topology is not a Baire space. Every point is closed, and the union of countably many points makes up the whole space.
- $\mathbb{Z}^+$ is a Baire space. Every subset is open, so the only subset with empty interior is the empty set.
- Every closed subspace of $\mathbb{R}$ is a complete metric space, hence a Baire space. In fact, the irrationals in $\mathbb{R}$ also form a Baire space. (See Exercise 6.)

**Lemma 1.** $X$ is a Baire space if and only if given any countable collection $\{U_n\}$ of open sets in $X$, each of which is dense in $X$, their intersection $\bigcap U_n$ is also dense in $X$.

**Proof.** The result follows easily from the facts that complements of open sets are closed sets (and vice versa) and $A \subseteq X$ is dense if and only if $X - A$ has empty interior.

Before proving that complete metric spaces are Baire spaces, we will need a simple result.

**Lemma 2.** If $X$ is a complete metric space, then $X$ is regular. Namely for each pair of a point $x$ and a closed set $B$ disjoint from $x$, there exist disjoint open sets containing $x$ and $B$, respectively.
Proof. Since $x \in B^c$ and $B^c$ is open, there is an open ball $B_\delta(x)$ of radius $\delta > 0$, centered at $x$, whose closure is contained in $B^c$. Then the open sets $B_\delta(x)$ and $X - B_\delta(x)$ give the result. \qed

Now we come to the first big result.

**Theorem 1** (Baire Category Theorem). If $X$ is a complete metric space, then $X$ is a Baire space.

**Proof.** Let $\{A_n\}$ be a countable collection of closed subsets of $X$, each with empty interior. We need to show that any non-empty open set $U_0$ contains a point not in any of the $A_n$.

Since $A_1$ does not contain $U_0$, choose $y \in U_0$ not in $A_1$. Since $X$ is regular and $A_1$ is closed, we can in fact choose a neighborhood $U_1$ of $y$ of diameter $< 1$ whose closure is contained in $U_0$ and disjoint from $A_1$. Now repeat by induction, where at the $n$th step we choose a point of $U_{n-1}$ not in $A_n$, and a neighborhood $U_n$ of that point with diameter $< 1/n$ and with closure contained in $U_{n-1}$ and disjoint from $A_n$.

Then we claim that the intersection $\bigcap U_n$ is non-empty, which will give the result. This is a result of the following lemma. \qed

**Lemma 3.** Let $C_1 \supseteq C_2 \supseteq \cdots$ be a nested sequence of nonempty closed sets in the complete metric space $X$. If $\text{diam } C_n \to 0$, then $\bigcap C_n \neq \emptyset$.

**Proof.** Choose $x_n \in C_n$ for each $n$. Then $\{x_n\}$ is Cauchy, and eventually lies inside each $C_k$, whence the limit $x$ is in each $C_k$ and thus in the intersection. \qed

Now we come to a crucial theorem.

**Theorem 2** (Uniform boundedness principle). Let $X$ be a Banach space and $Y$ be a normed vector space. Suppose $F$ is a collection of continuous (bounded) linear operators from $X$ to $Y$, such that for each $x \in X$, there is an $M_x \geq 0$ satisfying

$$\sup_{T \in F} \|Tx\| \leq M_x.$$  

Then

$$\sup_{T \in F} \|T\| < \infty.$$  

The gist of the theorem is that we want a sufficient condition to ensure the norms of a collection of operators are bounded. An obvious necessary condition is that the norms are pointwise bounded in the assumed sense, which turns out to be sufficient as well.

**Proof.** For $n \in \mathbb{Z}_{\geq 1}$, define

$$A_n := \left\{ x \in X : \sup_{T \in F} \|Tx\| \leq n \right\}.$$  

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Then $A_n$ is closed (since $X$ is complete and each $T \in F$ is continuous), contains 0, and is symmetric about the origin. Thus if some $A_n$ contains a non-empty open set, then it contains a ball of radius $\delta > 0$ centered at the origin, whence for any $x \in X$,

$$
\|Tx\| = \frac{2\|x\|}{\delta} \left| T \left( \frac{x}{\|x\|} \right) \right| \leq \frac{2\|x\|}{\delta} n,
$$

whence $\|T\| \leq 2n/\delta$, giving the result. So we may assume that each $A_n$ has empty interior.

But $X = \bigcup A_n$ by the pointwise boundedness assumption, so by the Baire category theorem, $X = \bigcup A_n$ has empty interior, contradiction. □

**Corollary 1.** If a sequence of bounded linear operators on a Banach space converges pointwise, then these pointwise limits define a bounded linear operator.