Analysis Qualifying Exam Solutions

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Chapter 1

Spring 2011

1.1 Real Analysis

A1. (a) $\ell^1(\mathbb{Z})$ is separable. A countable set whose finite linear combinations are dense is $\{e_n\}_{n \in \mathbb{Z}}$, where $e_n$ has a 1 in the $n$th position and is 0 everywhere else. If $x \in \ell^1(\mathbb{Z})$, then the sums $\sum_{k=-N}^{N} x_k e_k$ approximate $x$ arbitrarily well in the norm as $N \to \infty$ since $\sum_{k} |x_k| < \infty$.

(b) $\ell^\infty(\mathbb{Z})$ is not separable. This is proved by finding an uncountable number of disjoint open sets, whence every dense set must have an element in each interval and thus must be uncountable. Recalling that $\ell^\infty(\mathbb{Z})$ consists of all bounded sequences, consider all sequences containing just 0’s and 1’s. These map surjectively onto the real numbers by considering binary expansions, hence are uncountable. Any two distinct sequences of 0’s and 1’s have distance 1 from each other in the norm, so for any $r < \frac{1}{2}$, taking a ball of radius $r$ around each of these sequences gives an uncountable number of disjoint open sets.

A2. (a) If $f$ is measurable, then so is $|f|$. We need to show that $S(a) := |f|^{-1}((a, \infty))$ is measurable for each $a \in \mathbb{R}$. The case $a \leq 0$ is trivial since then $S(a) = \emptyset$ is measurable. Note that when $a > 0$, we have

$|f|^{-1}((\infty, a)) = f^{-1}((-a, a))$.

Now, $f$ measurable implies $f^{-1}((\infty, a))$ is measurable for each $a \in \mathbb{R}$, so each $f^{-1}([a, \infty))$ is measurable by taking complements. Therefore each $f^{-1}((a, \infty))$ is measurable since

$f^{-1}((a, \infty)) = \bigcup_{n=1}^{\infty} f^{-1}([a + \frac{1}{n}, \infty))$,

so $f^{-1}((-a, a)) = f^{-1}((\infty, a)) \cap f^{-1}((-a, \infty))$ is measurable, as desired.
(b) $|f|$ measurable does not imply that $f$ is measurable, as long as $X$ contains non-measurable sets. For if $\mathcal{N}$ is a non-measurable set, define

$$f(x) = \begin{cases} -1 & x \in \mathcal{N}; \\ 1 & \text{otherwise}. \end{cases}$$

Then $|f| \equiv 1$ is measurable, but $f^{-1}((-\infty,0)) = \mathcal{N}$ is not measurable.

A3. Haven’t reviewed Fourier series.

A4. $f \in L^1$ implies $\int |f| < \infty$, and $f$ uniformly continuous means that for any $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$. Suppose for contradiction that $x$ is a sequence with $|x_k| \to \infty$ for which $f(x_k) \not\to 0$. Then there is an $\epsilon > 0$ such that for each $N > 0$, there exists $n \geq N$ such that $|f(x_n)| \geq \epsilon$. Thus $x$ has a subsequence $\{x_{n_k}\}$ such that $|f(x_{n_k})| \geq \epsilon$ for each $k \geq 1$.

We may assume the $x_{n_k}$ are distinct, deleting repeats if necessary. Then by uniform continuity, choose $\delta > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{2}$ whenever $|x - y| < \delta$. Also, make $\delta$ small enough so that $|x_{n_k} - x_{n_j}| \geq \delta$ whenever $k \neq j$.

Then taking a ball of radius $\frac{\delta}{2}$ around each $x_k$ and letting $E$ be the union of these balls, we see that

$$\int |f| \geq \int_E |f| \geq \sum_{k=1}^{\infty} \delta \cdot \frac{\epsilon}{2} = \infty,$$

contradicting the fact that $f \in L^1$.

A5. Without (1), consider the sequence

$$\{f_n\} = \left\{ \chi_{[0,1]}, \frac{1}{2} \chi_{[0,2]}, \frac{1}{4} \chi_{[0,4]}, \ldots \right\},$$

which converges pointwise to the function $f \equiv 0$ for all $x$. Then (1) fails and

$$\lim_{n \to \infty} \int |f_n - f| = \lim_{n \to \infty} \int |f_n| = 1 \neq 0.$$

With (1), we can prove the result. Set $g_n = \inf \{f_n, f_{n+1}, \ldots\}$, which is measurable since the $f_n$ are measurable. Then $0 \leq g_n \leq f_n$ and $g_n \uparrow f$, so $\int |g_n - f| \to 0$ by the dominated convergence theorem, whence also $\int g_n \to \int f$.

Now we use (1) and the fact that $|f_n - g_n| = f_n - g_n$ to deduce that

$$\int |f_n - f| \leq \int |f_n - g_n| + \int |g_n - f| = \int f_n - \int g_n + \int |g_n - f| \to 0.$$
1.2 Complex Analysis

B6. $g$ is holomorphic on $\overline{U}$. Let $z_0 \in U$. Since $f$ is holomorphic at $z_0 \in U$, we can write $f(\bar{z}) = \sum_{n=0}^{\infty} a_n(\bar{z} - \bar{z}_0)^n$ for $z$ near $z_0$. Thus

$$g(z) := f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

is a power series expansion for $g$ near $z_0$.

B7. Since $|f(z)| \leq C|z|^k$ for $z$ outside $D_R$, all the poles of $f$ are inside $D_R$, and there can be only finitely many such poles since $D_R$ is compact. Let $\tilde{f}(z)$ be the sum of the principal parts of $f$ at each of these poles. Then $g := f - \tilde{f}$ is an entire function that still satisfies a bound of the form $|g(z)| \leq C'|z|^k$ for $|z| \geq R$ since the principal parts are bounded away from their poles.

We claim that $g$ is a polynomial of degree $\leq k$, which will imply that $f$ is a rational function. Since $g$ is entire, it suffices to prove that $g^{(j)}(0) = 0$ for all $j > k$ (look at the power series expansion). For each $R' \geq R > 0$, the Cauchy inequalities imply

$$|g^{(k+1)}(0)| \leq \frac{(k+1)!}{R^{k+1}} C'R^k \to 0,$$

as $R' \to \infty$, giving the desired result.

B8. (i) $\implies$ (ii): Suppose $a$ is a zero of $f$ of order $l \geq k$. Let $f(z) = (z-a)^l \sum_{n=0}^{\infty} a_n(z-a)^n$ be the power series expansion of $f$ near $a$, so that $a_0 \neq 0$. Let $C = |a_0|$ and choose $0 < \epsilon < 1$ sufficiently small such that $B_\epsilon(a) \subseteq U$ and $|\sum_{n=1}^{\infty} a_n \epsilon^n| \leq |a_0|$. Then for all $z \in B_\epsilon(a)$,

$$|f(z)| \leq |z-a|^l(|a_0| + |a_0|) \leq 2|a_0||z-a|^k.$$

(ii) $\implies$ (i): Suppose (ii) holds and let $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ be the power series expansion for $f$ near $a$. Then for all $z \in B_\epsilon(a)$ with $z \neq a$,

$$\left| \frac{f(z)}{(z-a)^k} \right| \leq C$$

implies $a_0 = a_1 = \cdots = a_{k-1} = 0$.

B9. The entire function $\sin z := \frac{1}{i \pi}(e^{iz} - e^{-iz})$ has zeroes at $\pi k$, $k \in \mathbb{Z}$, an essential singularity at $\infty$, and no other zeroes or singularities. The function $\sin \frac{1}{z}$ therefore has zeroes at $\frac{1}{\pi k}$ and an essential singularity at $0$, with no other zeroes or singularities.
Since \( \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \) we also have \( \frac{1}{z} \sin \frac{1}{z} = z^{-1} - \frac{z^{-3}}{3!} + \frac{z^{-5}}{5!} - \cdots \).

Thus the residue at 0 of the product of these two series, namely the coefficient of \( z^{-1} \) of the Laurent series, is 0.

Is this the right answer?

B10. Set

\[ f(z) = \frac{ze^{iz}}{z^2 + 2z + 10}. \]

The denominator factors into \((z + 3i + 1)(z - 3i + 1)\), so there are poles at \(3i - 1\) and \(-3i - 1\). We will integrate \( f \) over the upper-half-plane semicircle consisting of \( \gamma \), the real-axis interval \([-R, R]\), and the semicircle \( \gamma_R \) of radius \( R \). The imaginary part of the integral over \( \gamma \) gives the real integral we want to evaluate. Since we are taking \( R \) large, we pick up the residue at \( 3i - 1 \), which is

\[
\lim_{z \to 3i - 1} \frac{ze^{iz}}{(z + 3i + 1)(z - 3i + 1)} = \frac{(3i - 1)e^{-3-i}}{6i} = \frac{1}{6e^3} \left( 3 \cos 1 + \sin 1 + i(\cos 1 - 3 \sin 1) \right).
\]

What a mess! The integral over \( \gamma_R \) vanishes thanks to the \( 1/z \) decay of \( f \) near the real axis, together with the better decay of \( f \) due to \( e^{iz} \) away from the real axis.
Chapter 2

Fall 2010

2.1 Real Analysis

A1. Yes, the fact that \( f^{-1}\left((r, \infty)\right) \) is measurable for each \( r \in \mathbb{Q} \) implies that \( f \) is measurable. Taking complements, we see that \( f^{-1}\left((\infty, r]\right) \) is measurable for each \( r \in \mathbb{Q} \). Now for any \( s \in \mathbb{R} \), let \( r_k \) be an increasing sequence of rational numbers \( < s \) such that \( r_k \to s \). Then

\[
\int f^{-1}\left((\infty, s]\right) = \bigcup_{k=1}^{\infty} f^{-1}\left((\infty, r_k]\right),
\]

so \( f \) is measurable.

A2. Suppose \( 1 \leq p \leq q \leq \infty \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \), where we allow \( p = 1, q = \infty \). Since \( f, f_n \in L^p \) and \( g, g_n \in L^q \), we also have \( f - f_n \in L^p \) and \( g - g_n \in L^q \). Hölder's inequality implies \( f_n g_n \) and \( fg \) are in \( L^1 \). Note that

\[
||fg| - |f_n g_n|| = ||fg| - |f_n g| + |f_n g| - |f_n g_n|| \leq |g||f - f_n| + |f_n||g - g_n|.
\]

Thus using Hölder's inequality,

\[
\int ||fg| - |f_n g_n|| \leq \int |g||f - f_n| + \int |f_n||g - g_n| = ||g||_q ||f - f_n||_1 + ||f_n||_p ||g - g_n||_q.
\]

Since \( ||g||_q \) and \( ||f_n||_p \) are bounded but \( ||f - f_n||_p \to 0 \) and \( ||g - g_n||_q \to 0 \), we see that \( ||fg - f_n g_n||_1 \to 0 \), which gives the desired result since

\[
\int |fg| - \int |f_n g_n| \leq \int ||fg| - |f_n g_n|| \to 0.
\]
(Maybe I’m missing something, but I don’t think the case \( p = 1, q = \infty \) is any different than \( p > 1 \).)

**A3.** Let \( H \xrightarrow{T_n} \mathbb{C} \) denote the bounded linear functional defined by \( T_n(y) = (y, x_n) \). Then the countable collection \( \{T_n\} \) is pointwise bounded by assumption, hence the norms are bounded by the uniform boundedness property. So there exists \( M \geq 0 \) such that 
\[
|\langle y, x_n \rangle| \leq M \|y\| \quad \text{for all } y \in H.
\]
In particular, choosing \( y = x_n \) gives \( \|x_n\| \leq M \), establishing the desired result.

**A4.** Suppose \( f \) is measurable such that \( \int |f|^p < \infty \) and \( \int |f|^r < \infty \). Set
\[
E = \{x \in X : |f(x)| \geq 1\},
\]
which is measurable. Then
\[
\int_X |f|^q = \int_E |f|^q + \int_{X-E} |f|^q \leq \int_E |f|^p + \int_{X-E} |f|^r \leq \int_X |f|^p + \int_X |f|^r < \infty,
\]
so \( f \in L^q \) as desired.

**A5.** I will only sketch the proof, which is quite long (see Stein 175-177 for details). If \( x \in M \), then take \( y = x \). Otherwise, set \( d := \inf_{y \in M} \|x - y\| \). Since \( M \) is closed and \( x /\in M \), \( d > 0 \). Now choose a sequence \( \{y_n\} \) in \( M \) such that \( \|x - y_n\| \to d \), which we will prove is Cauchy, and whose limit in \( M \) (since \( M \) is closed) will be the desired unique element \( y \in M \) closest to \( x \). For
\[
\|x - y\| \leq \|x - y_n\| + \|y_n - y\| \to d \quad \text{as } n \to \infty,
\]
and uniqueness is shown by first proving that \( x - y \) is orthogonal to \( M \) (take perturbations of \( y \) in \( M \) and look at the norm of the difference with \( x \)) and then applying the Pythagorean theorem to \( x - y \) and \( y - y' \), where \( y' \in M \) is another element minimizing distance to \( x \).

To show the sequence is Cauchy, we use the parallelogram law
\[
\|A + B\|^2 + \|A - B\|^2 = 2 (\|A\|^2 + \|B\|^2) \quad \text{for all } A, B \in H
\]
with \( A = x - y_n \) and \( B = x - y_m \).
2.2 Complex Analysis

B6. We write

\[ f(z) = 1 - \cos z = 1 - \frac{e^{iz} + e^{-iz}}{2} = \left( e^{iz} - e^{-iz} \right)^2, \]

so the zeroes of \( f \) are at \( z = 2\pi \cdot k \) for \( k \in \mathbb{Z} \) and are each of order 2 since

\[ \lim_{z \to 0} \frac{e^{iz} - 1}{z} = \lim_{z \to 0} \frac{ie^{iz}}{1} = 1 \]

by L’Hospital’s rule (or apply L’Hospital to \( \lim_{z \to 2\pi k} (1 - \cos z)/(z - 2\pi k)^2 \) and note that you get a constant).

B7. (i) We have

\[ f(z) = \frac{1}{2i} \left( e^{\frac{1+i}{n}} - e^{-\frac{1+i}{n}} \right). \]

Clearly the only singularity of \( f \) is at \( z = -1 \), and we claim that the singularity

is essential. Consider the sequence \( \{x_n\} := \{-1 + \frac{i}{n}\} \). Then \( x_n \to -1 \) and

\[ |f(x_n)| = \left| \frac{1}{2i} \left( e^{-n+i} - e^{-n-i} \right) \right| \geq \frac{e^n}{4} \to \infty. \]

On the other hand, let \( \{y_n\} := \{-1 + \frac{1}{n}\} \), which also satisfies \( y_n \to -1 \), but

\[ |f(y_n)| = \left| \frac{1}{2i} \left( e^{i(-n+1)} - e^{-i(-n+1)} \right) \right| \leq 1. \]

So \( z = -1 \) is an essential singularity of \( f \).

(ii) Recall that

\[ \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \]

and

\[ \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots, \]

as well as the identity

\[ \sin(a + b) = \sin a \cos b + \cos a \sin b, \]

which holds for all \( a, b \in \mathbb{C} \) (since the proof relies on the properties of exponentials). So

\[
\sin \frac{z}{z+1} = \sin \left( 1 - \frac{1}{z+1} \right) \\
= \sin 1 \cos \frac{1}{z+1} - \cos 1 \sin \frac{1}{z+1} \\
= \sin 1 \left( 1 - \frac{1}{2!(z+1)^2} + \cdots \right) - \cos 1 \left( \frac{1}{z+1} - \frac{1}{3!(z+1)^3} + \cdots \right)
\]
is the Laurent series for $f(z)$.

(iii) The residue of $f$ at $z = -1$ is the coefficient of $\frac{1}{z+1}$ in the Laurent series, namely $-\cos 1$.

**B8.** We integrate

$$f(z) = \frac{ze^{iz}}{(z+3i-1)(z-3i-1)}$$
on a semicircle of radius $R$ in the upper half plane, picking up the residue at $z = 3i + 1$. The real part of the integral along the real axis goes to the desired integral as $R \to \infty$, while the integral along the curved portion goes to 0 by the decay of $e^{iz}$ away from the real axis. The residue is

$$\text{res}_{3i+1} f = \lim_{z \to 3i+1} (z-3i-1)f(z) = \frac{1}{6e^3}(3 \cos 1 + \sin 1 + i(-\cos 1 + 3 \sin 1)).$$
The real part of $2\pi i \cdot \text{res}_{3i+1} f$, which is the value of the desired integral, is therefore $\frac{\pi}{3e^3}(\cos 1 - 3 \sin 1)$.

**B9. (i)** If $f$ is entire and $|f(z)| \leq C|z|^n$ for all $z \in \mathbb{C}$, then $f$ is a polynomial of degree $\leq n$. To see this, it suffices to prove that $f^{(k)}(0) = 0$ for all $k > n$ (look at the power series). This in turn can be proved by the Cauchy inequality: for any $R > 0$ we have

$$|f^{(k)}(0)| \leq \frac{(k)!}{R^k} CR^n \to 0$$
as $R \to \infty$ when $k > n$.

(ii) The dimension is $n + 1$ since a basis is given by $\{1, z, z^2, \ldots, z^n\}$.

**B10.** The polynomials $f(z) := 2z^5 - z^3 + 3z^2 - z$ and $g(z) = 8$ are both entire and have no zeroes on the circle $|z| = 1$. Indeed, $|f(z)| \leq 2 + 1 + 3 + 1 = 7$ on this circle, while $|g(z)| = 8$ on the circle. Thus by Rouche’s theorem, the number of zeroes of $f + g$ inside the circle is the same as the number of zeroes of $g$ inside the circle, namely zero. Thus $f + g$ has all 5 of its zeroes (counted with multiplicities) outside the circle.
Chapter 3

Spring 2010

3.1 Real Analysis

1. (a) Since the sequence is Cauchy, we can pick a subsequence \( \{f_{n_k}\} \) such that
\[
\|f_{n_k} - f_{n_k+1}\|_p < \frac{1}{2^k}
\]
for each \( k \geq 1 \). We claim that this subsequence converges pointwise almost everywhere. To prove this, we consider the series
\[
f := f_{n_1} + \sum_{k=1}^{\infty} (f_{n_{k+1}} - f_{n_k})
\]
and
\[
g := |f_{n_1}| + \sum_{k=1}^{\infty} |f_{n_{k+1}} - f_{n_k}|
\]
with partial sums \( S_K(f) \) and \( S_K(g) \), respectively. Then the triangle inequality implies
\[
\|S_K(g)\|_p \leq \|f_{n_1}\|_p + \sum_{k=1}^{K} \|f_{n_{k+1}} - f_{n_k}\|_p < \|f_{n_1}\|_p + 1,
\]
and we have \( |S_K(g)|^p \to |g|^p \), so the monotone convergence theorem implies
\[
\int |g|^p = \lim_{K \to \infty} \int |S_K(g)|^p < \infty.
\]
So \( g \in L^p([0, 1]) \), and since \( |f| \leq g \), we have \( f \in L^p([0, 1]) \) as well. In particular, the series defining \( f \) converges (is less than infinity) almost everywhere. By construction, \( S_{K-1}(f) = f_{n_{k+1}} \), so \( f_{n_k}(x) \to f(x) \) for almost every \( x \). (See also Stein 159-60.)
(b) Consider the sequence \( \{ \chi_{[0,1]}, \chi_{[0,\frac{1}{2}]} , \chi_{[\frac{1}{2},1]} , \chi_{[0,\frac{1}{2}]} , \ldots \} \). The sequence is Cauchy and converges to the 0 function in the norm. But the pointwise limit fails to exist anywhere since every \( x \in [0,1] \) occurs in infinitely many intervals of the form \([\frac{m}{2^n}, \frac{m+1}{2^n}]\).

2. (a) The key thing is to show that if \( \{ f_n \} \to 0 \) in \( L^p \), then the same is true in \( L^q \) (continuity). Here we must use the fact that \( L^p \subset L^q \); it is not true in general (e.g. look at \( \mathbb{R} \)). I don’t know how to do this.

(b) Suppose for contradiction that \( \inf_{E \in \mathcal{M}} \mu(E') = 0 \), so that we can choose \( E_1, E_2, \ldots \) in \( \mathcal{M}' \) with \( \mu(E_n) \to 0 \). Then set \( f_n = \mu(E_n)^{-1/p} \chi_{E_n} \), which is simple and hence measurable, so that \( \| f_n \|_p = 1 \), but \( \| f_n \|_q = \left[ \mu(E_n)^{1-q/p} \right]^{1/q} = \mu(E_n)^{-\epsilon} \) for some \( \epsilon > 0 \). But this means that \( \| f_n \|_q \to \infty \), contradicting the boundedness from (a).

3. Note that \( \pi_V \) is the map that sends \( x \in H \) to the unique closest element in \( V \) in the norm of \( H \). One shows that \( x - \pi_V(x) \perp V \).

(a) For any \( x \in H \), \( x - \pi_V(x) \perp V \), so by the Pythagorean theorem
\[
\|x\|^2 = \|(x - \pi_V(x)) + \pi_V(x)\|^2 = \|x - \pi_V(x)\|^2 + \|\pi_V(x)\|^2.
\]
Thus \( \|\pi_V(x)\| \leq \|x\| \), whence \( \|\pi_V\| \leq 1 \). It follows that \( \|\pi_V\| = 1 \) since \( \|\pi_V(x)\| = \|x\| \) for nonzero \( x \in V \).

\( \pi_V \) is the identity on \( V \) since the unique closest element in \( V \) of an element in \( V \) itself. Thus \( \pi_V \) is idempotent.

We need to show that \( (\pi_V x, y) = (x, \pi_V y) \) for all \( x, y \in H \). This follows easily from
\[
0 = (x - \pi_V x, \pi_V y) = (x, \pi_V y) - (\pi_V x, \pi_V y)
\]
and likewise
\[
0 = (\pi_V x, y - \pi_V y) = (\pi_V x, y) - (\pi_V x, \pi_V y).
\]

(b) Set \( V := \text{im}(P) \), which is a subspace since it is the image of a linear operator, and which we claim is closed. For if \( \{ x_n \} \) is a Cauchy sequence in \( V \), with limit \( y \in H \), then \( P \) idempotent and continuous implies \( \{ x_n \} = \{ Px_n \} \to Py \), whence \( Py = y \) proves that \( y \in V \). So \( V \) is closed.

Using the fact that \( P \) and \( \pi_V \) are self-adjoint,
\[
(Px, y) = (x, Py) = (x, \pi_V Py) = (\pi_V x, Py) = (P \pi_V x, y) = (\pi_V x, y)
\]
for all \( x, y \in H \). Thus \( (P - \pi_V)x, y) = 0 \) for all \( x, y \), so \( P - \pi_V \) is perpendicular to all of \( H \), hence 0, and this holds for each \( x \), whence \( P = \pi_V \).
4. (a) The map \( \ell^1 \xrightarrow{\Lambda} (\ell^\infty)^* \) defined by

\[
\Lambda(\xi)(\eta_1, \eta_2, \ldots) = \sum_i \xi_i \eta_i
\]

is well defined since

\[
|\Lambda(\xi)(\eta)| \leq \sum_i |\xi_i| |\eta_i| \leq \|\eta\|_{\infty} \sum_i |\xi_i| = \|\eta\|_{\infty} \|\xi\|_1 < \infty.
\]

This also shows that the operator norm \( \|\Lambda(\xi)\| \leq \|\xi\|_1 \), which is in fact equality since if \( \eta = (\xi_1/|\xi_1|, \xi_2/|\xi_2|, \ldots) \), which has \( \|\eta\|_{\infty} = 1 \), then

\[
|\Lambda(\xi)(\eta)| = \left| \sum_i \xi_i \xi_i/|\xi_i| \right| = \left| \sum_i |\xi_i| \right| = \|\xi\|_1.
\]

So \( \Lambda \) is norm-preserving. It is clearly injective since if \( \xi \neq 0 \), then some \( \xi_i \neq 0 \), hence the corresponding \( \eta \) with a 1 in the \( i \)th position and 0’s everywhere else satisfies \( \Lambda(\xi)(\eta) = \xi_i \neq 0 \), hence \( \Lambda(\xi) \neq 0 \).

For the failure of surjectivity, I need to show that \( (\ell^\infty)^* \) contains functionals that vanish on all \( \eta \) such that \( \eta_n \to 0 \) as \( n \to \infty \). But I don’t know how to find such a functional. Perhaps as some sort of limit? Heeeeelp!!!

(b) We need \( \frac{1}{p} + \frac{1}{q} = 1 \), but I’m not sure about other conditions.

5. Suppose for contradiction that there is a set \( E \) such that the subset \( \Sigma := \{ x \in E: f^n(x) \notin E \text{ for all } n \geq 1 \} \) has positive. We will prove that

\[
\Sigma, f(\Sigma), \ldots, f^n(\Sigma)
\]

are almost disjoint (their pairwise intersections have measure 0) for each \( n \geq 1 \), which will contradict the fact that \( \mu(X) < \infty \) since then the fact that \( f \) is measure preserving will imply that

\[
\mu(\Sigma \cup f(\Sigma) \cup \cdots \cup f^n(\Sigma)) = (n + 1)\mu(\Sigma) \to \infty.
\]

(Note that if \( \mu(X) = \infty \), for instance \( X = \mathbb{R} \) with the Lebesgue measure, then \( f \) can be a shift by a constant and the claim fails.)

We use induction. When \( n = 0 \), the claim is trivial. Now suppose for induction that for some \( k \geq 0 \),

\[
\Sigma, f(\Sigma), \ldots, f^k(\Sigma)
\]

are almost disjoint. Since \( f \) is measure preserving,

\[
f(\Sigma), f^2(\Sigma), \ldots, f^{k+1}(\Sigma)
\]

are almost disjoint.
must then also be almost disjoint. But Σ has empty intersection with each 
f_j(Σ) by the definition of Σ, whence

\[ Σ, f(Σ), f^2(Σ), \ldots, f^{k+1}(Σ) \]

are almost disjoint.

### 3.2 Complex Analysis

6. (a) The singularities of \( f \) inside the circle of radius 2 are simple poles at \( z = \pm 1 \). The corresponding residues are

\[
\text{res}_1(f) = \lim_{z \to 1} (z - 1) \cdot f = 1
\]

and

\[
\text{res}_{-1}(f) = \lim_{z \to -1} (z + 1) \cdot f = 1.
\]

Thus \( \int f = 2\pi i (1 + 1) = 4\pi i \).

(b) In the annulus \( 1 < |z| < 3 \), we have

\[
\frac{1}{z + 1} = \frac{1/z}{1 + 1/z} = \frac{1}{z} \left( 1 - \frac{1}{z} + \frac{1}{z^2} - \cdots \right);
\]

\[
\frac{1}{z - 1} = \frac{1/z}{1 - 1/z} = \frac{1}{z} \left( 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots \right);
\]

\[
\frac{1}{z + 3} = \frac{1/3}{1 + z/3} = \frac{1}{3} \left( 1 - \frac{z}{3} + \frac{z^2}{9} - \cdots \right).
\]

Expanding \( f \) by partial fractions, we get

\[
f(z) = \frac{6z + 2}{(z^2 - 1)(z + 3)} = \frac{1}{z + 1} + \frac{1}{z - 1} - \frac{2}{z + 3}.
\]

so the coefficient of \( \frac{1}{z} \) of the Laurent series of \( f \) is \( 1 + 1 = 2 \), which agrees with the sum of the residues in (a). To check the integral, note that every term of the Laurent series has a primitive (and hence its integral vanishes) except the one corresponding to \( \frac{1}{z} \), and integrating \( \frac{1}{z} \) over any circle centered at the origin gives \( 2\pi i \). Multiplying by the coefficient, namely the residue, yields \( 4\pi i \).

7. Skip.
8. (a) If \( f \) is analytic and injective on \( U \), then \( f' \) is nonzero on \( U \). Assume for contradiction that \( f'(z_0) = 0 \). Let the power series for \( f \) about \( z_0 \) be

\[
 f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots .
\]

Then \( a_1 = 0 \) since the derivative vanishes, so

\[
 f(z) - a_0 = a_k(z - z_0)^k + G(z),
\]

where \( a_k \neq 0, k \geq 2 \), and \( G \) vanishes to order at least \( k + 1 \) at \( z_0 \). Now for small \( w \) write

\[
 f(z) - a_0 - w = F(z) + G(z), \quad \text{where } F(z) := a_k(z - z_0)^k - w.
\]

The idea is that we can choose a small enough circle \( C \) around \( z_0 \) and \( w \) small enough so that \( |F(z)| > |G(z)| \) on \( C \) and \( F \) has \( k \) roots inside \( C \). Then by Rouche’s theorem, \( F(z) + G(z) \) has \( k \) roots inside \( C \), contradicting the injectivity of \( f \).

One way to pick \( C \) and \( w \) is the following. By the vanishing of \( G \), choose \( \epsilon > 0 \) so small that \( |G(z)| < |a_k| \frac{1}{2} \epsilon^k \) on the circle \( C_\epsilon(z_0) \). Then choose \( 0 < \delta < \frac{1}{2} \epsilon^k \) and set \( w = \delta a_k \), so that

\[
 |F(z)| = |a_k||(z - z_0)^k - \delta| \geq |a_k|\frac{1}{2} \epsilon^k > |G(z)|
\]

on \( C_\epsilon(z_0) \). Then \( F \) has its zeroes inside \( C_\epsilon(z_0) \) since \( \delta^{1/k} < \epsilon \), so we are done.

(b) \( f' \) never 0 on \( U \) does not ensure that \( f \) is injective on \( U \). Let \( U = \mathbb{C} \) and \( f(z) = e^z \). Then \( f'(z) = e^z \) has no roots, but \( e^0 = e^{2\pi i} \), so \( f \) is not injective.

Another example is \( f(z) = z^2 \), on the punctured disc centered at the origin.

9. There is a mistake: the upper half disc should be defined as all \( z \) with \( |z| < 2 \) and \( \text{Im } z \geq 0 \). First scale this map by \( \frac{1}{2} \), then map to the upper half plane by \( z \mapsto -(z + \frac{1}{2}) \), and finally map to the unit disc with \( z \mapsto \frac{1}{1 + z^2} \). All automorphisms of the unit disc are rotations composed with Blaschke factors. Chasing \( i \), we see that \( i \mapsto \frac{i}{2} \mapsto \frac{3i}{2} \mapsto \frac{1}{3i} \). So to map \( \frac{1}{3} \) to 0, we use the Blaschke factor \( \psi_\alpha(z) = \frac{a - z}{1 - \alpha z} \), with \( \alpha = \frac{1}{3} \). Thereafter we can compose with arbitrary rotations since they fix the origin.

10. Skip.
Chapter 4

Fall 2009

4.1 Real Analysis

1. All the $f_n, f$ are integrable since they are bounded by $f_1$, which is integrable. By assumption

$$\int |f_n - f| = \int (f_n - f) = \int f_n - \int f \to 0.$$ 

Set

$$E_\delta = \{ x \in X : f_n(x) - f(x) \geq \delta \text{ for all } n \},$$

which is measurable since it is the intersection $\bigcap_{n}(f_n - f)^{-1}([\delta, \infty])$. Let $E$ be the set of all $x \in X$ for which $f_n(x) \not\to f(x)$, which is measurable since it is the union $\bigcup_k E_{1/k}$. Then

$$\int |f_n - f| \geq \int_{E_{1/k}} |f_n - f| \geq \frac{m(E_{1/k})}{k}$$

for all $n$, so $m(E_{1/k}) = 0$. Thus $m(E) = 0$ as well.

If we don’t assume $f_1$ is integrable, then all bets are off. For instance, if $X$ is a set with infinite measure (e.g. $\mathbb{R}$), $f_n \equiv 2$ for all $n$, and $f \equiv 1$, then $\int f_n \to \int f$ since both sides are infinite, but $f_n \not\to f$ for all $x$.

2. For boundedness, simply note that

$$|\hat{f}(x)| \leq \int_{-\infty}^{\infty} |f(y)| dy = \|f\|_1 < \infty.$$ 

For continuity, note that $f(x)e^{-2\pi ixy} \to f(x)e^{-2\pi ix_0y}$ for all $y \in \mathbb{R}$ as $x \to x_0$ by the continuity of the exponential, so $\hat{f}(x) \to \hat{f}(x_0)$ by the dominated convergence theorem.
3. The function $H \xrightarrow{L} \mathbb{C}$ sending $y \mapsto \lim_{n \to \infty} (x_n, y) = \lim_{n \to \infty} (y, x_n)$ is well-defined and linear. We will show that $f$ is bounded/continuous. Then by the Riesz representation theorem, there exists $x \in H$ such that $f(y) = (y, x)$ for all $y \in H$. Thus

$$
\lim_{n \to \infty} (x_n, y) = \lim_{n \to \infty} \left( \frac{y}{x_n} \right) = \lim_{n \to \infty} (x, y)
$$

gives the desired result.

To prove boundedness, define the collection of bounded linear functionals $H \xrightarrow{T} \mathbb{C}$ by

$$
T_n y = (y, x_n).
$$

By assumption, $\sup_n |T_n y| < \infty$ for each $y$, so by the uniform boundedness principle, $\sup_n \|T_n\| < \infty$. Thus there is some $M \geq 0$ such that $|T_n y| \leq M\|y\|$ for all $y \in H$, whence

$$
|f(y)| = \left| \lim_{n \to \infty} (y, x_n) \right| \leq \lim_{n \to \infty} |(y, x_n)| \leq M\|y\|
$$

proves that $f$ is bounded.

4. Since $f \in L^\infty(X)$, $|f| \leq \|f\|_\infty$ a.e. Thus

$$
\left| \int fg \right| \leq \int |fg| \leq \|f\|_\infty \int |g|
$$

by monotonicity and the fact that we can ignore a set of measure 0 when integrating. This proves that $T$ is bounded and that $\|T\| \leq \|f\|_\infty$. Next we will prove that $\|T\| \geq \|f\|_\infty$. We need to show that given $\epsilon > 0$, there is a $g$ such that

$$
\left| \int fg \right| > (\|f\|_\infty - \epsilon) \int |g|
$$

by definition of $\|f\|_\infty$, the set

$$
E_\delta := \{ x \in X : |f(x)| > \|f\|_\infty - \delta \}, \quad \delta > 0
$$

has positive measure, and it is measurable since $f$ is measurable. Set $g(x) := \chi_{E_\delta}(x)|f(x)|/|f(x)|$, which is in $L^1(X)$ since $m(E_\delta) \leq m(X) < \infty$. Then

$$
\left| \int fg \right| = \int_{E_\delta} |f| > m(E_\delta)(\|f\|_\infty - \delta) = (\|f\|_\infty - \delta) \int |g|,
$$

so taking $\delta = \epsilon$ gives the desired result. Note that the strict inequality depends on the fact that $m(E_\delta) > 0$. 

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5. The claim is false. Consider the sequence of intervals

\[ \{I_n\} := \left\{ [0, 1], \left[ 0, \frac{1}{2} \right], \left[ \frac{1}{2}, 1 \right], \left[ 0, \frac{1}{3} \right], \left[ \frac{1}{3}, \frac{2}{3} \right], \left[ \frac{2}{3}, 1 \right], \left[ 0, \frac{1}{4} \right], \ldots \right\} \]

and let \( \{\chi_{I_n}\} \) be the corresponding sequence of characteristic functions. Then \( \|\chi_{I_n}\|_2 = \sqrt{\mu(I_n)} \to 0 \) but the sequence \( \{\chi_{I_n}(x)\} \) has infinitely many 0’s and 1’s for each \( x \), and therefore does not converge.

4.2 Complex Analysis

6. We prove Schwarz’s lemma.

**Lemma 1** (Schwarz’s lemma). Let \( \Delta \xrightarrow{f} \Delta \) be holomorphic such that \( f(0) = 0 \), where \( \Delta \) is the open unit disc. Then

(i) \( |f(z)| \leq |z| \) for all \( z \in \Delta \);

(ii) if equality holds in (i) for some \( z_0 \neq 0 \), then \( f \) is a rotation;

(iii) \( |f'(0)| \leq 1 \), and if equality holds then \( f \) is a rotation.

**Proof.** (i): Since \( f(0) = 0 \), \( f(z)/z \) is holomorphic in \( \Delta \). If \( |z| = r < 1 \), we have

\[ \left| \frac{f(z)}{z} \right| \leq \frac{1}{r}, \]

and by maximum modulus this \( \frac{1}{r} \) bound holds for all \( |z| \leq r \). Letting \( r \to 1 \) shows that 1 is an upper bound for all \( z \in \Delta \).

(ii): By assumption \( |f(z)| = |z| \), so by maximum modulus \( f \) is constant. Thus \( f(z) = cz \), where \( |c| = 1 \), namely \( f \) is a rotation.

(iii): Write \( g(z) := f(z)/z \). Then

\[ g(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = f'(0), \]

and we already showed in (i) that \( g \) is bounded by 1 in absolute value. If \( |g(0)| = 1 \), then by (ii) we see that \( f \) is a rotation. \( \square \)

7. We will compute the Laurent series of \( f(z) = (1 + z^2)e^{1/z} \) to find the residue of \( f \) at the essential singularity \( z = 0 \). Since

\[ e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots, \]

\[ e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots, \]

\[ (1 + z^2)e^{1/z} = 1 + z + \frac{1}{2!z} + \frac{1}{3!z^2} + \cdots. \]
we have
\[ e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots. \]
Thus
\[ f(z) = \cdots + \left( \frac{1}{2!} + \frac{1}{4!} \right) \frac{1}{z^2} + \left( 1 + \frac{1}{3!} \right) \frac{1}{z} + \left( 1 + \frac{1}{2!} \right) + z + z^2, \]
so by the residue theorem the integral of \( f \) on the ccw. unit circle is
\[ 2\pi i \left( 1 + \frac{1}{3!} \right) = \frac{7\pi i}{3}. \]

8. We begin by proving that \( f \) is also holomorphic on the real axis. By Morera’s theorem, it is enough to check that the integral of \( g \) over every triangle is 0. If the triangle \( T \) has a side on the real axis, then slightly shifting it ensures that \( T \) lies entirely in one of the open half planes, hence the integral over an arbitrarily close contour is 0. Continuity then ensures the integral over \( T \) is 0. Similarly, if the triangle crosses the real axis, then splitting the triangle into smaller triangles and shifting them slightly gives the result. So \( f \) is entire.

Now define a new function
\[ g(z) := \begin{cases} f(z) & \text{Im}(z) \geq 0; \\ \overline{f(z)} & \text{otherwise}. \end{cases} \]
Then \( g \) is holomorphic on the open set \( \text{Im}(z) < 0 \) since for \( z \) near \( z_0 \) in the lower half plane we can use the power series expansion for \( f \) in the upper half plane to write
\[ f(z) = \sum a_n (z - z_0)^n, \]
whence
\[ g(z) = \overline{f(z)} = \sum \overline{a_n} (z - z_0)^n \]
is a power series expansion for \( g \). So \( g \) is holomorphic except possibly on the real axis, and continuous on the real axis since \( f \) is continuous and real-valued there. Replacing \( f \) by \( g \) in the above argument thus shows that \( g \) is entire. Since \( f \) and \( g \) agree on the entire upper half plane, they must agree everywhere by the uniqueness of analytic continuation.


10. Let \( F(z) = z^4 \) and \( G(z) = -6z - 3 \). Then for all \( |z| = 2 \), we have
\[ |F(z)| = 16 > 15 = 6 \cdot 2 + 3 \geq |G(z)|, \]
whence by Rouche’s theorem the functions $F$ and $F + G$ have the same number of solutions within the unit circle, namely all 4.
Chapter 5

Spring 2009

5.1 Real Analysis

1. There are typos in the problem, which I assume is meant to be the following. Let \( \{x_n\} \) be a sequence of measurable functions with \( 0 \leq f_n \leq 1 \) for each \( n \) such that \( \int_0^1 f_n \to 0 \). Then is it necessarily true that \( f_n \to 0 \) a.e.?

   No, this is false. Consider the sequence of intervals
   \[ \{I_n\} := \left\{ [0, 1], \left[ 0, \frac{1}{2} \right], \left[ \frac{1}{2}, 1 \right], \left[ 0, \frac{1}{3} \right], \left[ \frac{1}{3}, \frac{2}{3} \right], \left[ \frac{2}{3}, 1 \right], \left[ 0, \frac{1}{4} \right], \ldots \right\} \]
   and let \( \chi_{I_n} \) be the corresponding characteristic functions. Then the assumptions hold but \( \chi_{I_n} \to 0 \) nowhere.

2. If \( f \in L^\infty \), then for each \( p \geq 1 \),

   \[ \|f\|_p = \int |f|^p \leq \int \|f\|_\infty^p \leq \|f\|_\infty^p \]

   since \( \mu(X) = 1 \). Thus \( f \in L^p \) for each \( p \).

   For any \( \epsilon > 0 \), we wish to show that there is an \( N \) such that for all \( p \geq N \),
   \[ \|f\|_\infty - \|f\|_p < \epsilon \]. Choose \( \delta \) such that \( \epsilon > \delta > 0 \) and set \( E_\delta := \{x \in X : |f(x)| > \|f\|_\infty - \delta\} \), which has positive measure. Then

   \[ \|f\|_p^p \geq \int_{E_\delta} |f|^p \geq (\|f\|_\infty - \delta)^p \mu(E_\delta), \]

   which implies

   \[ \|f\|_p \geq (\|f\|_\infty - \delta)\mu(E_\delta)^{1/p}. \]

   Since \( \mu(E_\delta) \) has positive measure \( \leq 1 \), \( \mu(E_\delta)^{1/p} \to 1 \) as \( p \to \infty \), whence we get the desired result since \( \delta < \epsilon \).
3. First, we may assume $T$ is injective. (Assuming the axiom of choice, we can extend a basis of $\ker(T)$ to a basis for $X$, obtaining a decomposition $X = \ker(T) \oplus X'$. $T|_{X'}$ is a compact bijective linear map from $X'$ to $Y$.) Suppose for contradiction that $Y$ is infinite dimensional. Let $y_1, y_2, \ldots$ be a normal basis of $Y$. Since $T$ is bijective, there exist unique $x_1, x_2, \ldots$ such that $Tx_n = y_n$. Then $\{x_n\}$ cannot be bounded since if it were, compactness of $T$ would ensure the existence of a convergent subsequence $\{Tx_{n_k}\}$, which is impossible. So we may assume $|x_n| \to \infty$

This basis method does not seem so good. I can’t even prove that the $y_k$s don’t converge!

Help!

4. Define bounded linear functionals $H \xrightarrow{T_n} \mathbb{C}$ by $T_n y = \langle y, v_n \rangle$ for all $y \in H$. By assumption, this collection of linear functionals is bounded for each $y \in H$, hence bounded in the norm by the uniform boundedness principle. Thus there exists $M \geq 0$ such that $|T_n y| \leq M \|y\|$ for all $n, y$. In particular, $\|v_n\|^2 = |T_n v_n| \leq M \|v_n\|$ implies $\|v_n\| \leq M$ for all $n$.

5. Skip.

5.2 Complex Analysis

6. Since $f(z) \sin z \to 1$ as $z \to 0$ and $\sin z = z - z^3/3! + \cdots$ has a zero of order 1 at 0, the limit implies that $f$ has a pole of order 1 at 0, with residue 1. Thus the residue theorem shows that the integral of $f$ on the ccw. unit circle is $2\pi i$.

7. Since $f$ is bounded by $B$ outside $D_R$, all its poles must be in $D_R$, which is compact, so there can be only finitely many poles, at $a_1, \ldots, a_n$. Then $g(z) = f(z)(z - a_1) \cdots (z - a_n)$ is entire and satisfies

$$|g(z)| \leq C|z|^n$$

for some constant $C \geq 0$. We claim that this implies that $g$ is a polynomial of degree $\leq n$, for which it suffices to show that $g^{(k)}(0) = 0$ for all $k > n$ (look at the power series). This is proved by using the Cauchy inequalities:

$$|g^{(k)}(0)| \leq \frac{k!}{R^k} C R^n \to 0$$
as $R \to \infty$ when $k > n$. Thus
\[
f(z) = \frac{g(z)}{(z - a_1) \cdots (z - a_n)}
\]
expresses $f$ as a rational function.

8. Fix a point $z_0 \in \Omega$. Define $F(z) = \int_\gamma f(z) \, dz$, where $\gamma$ is any curve consisting of line segments parallel to the axes, starting at $z_0$ and ending at $z$. This is well-defined since our assumption ensures that the definition is independent of the curve chosen. We now prove that $F$ is holomorphic, with derivative $f$.

For $h$ so small that $z + h \in \Omega$, note that
\[
F(z + h) - F(z) = \int_\xi f(w) \, dw,
\]
where $\xi$ is any curve from $z$ to $z + h$ composed of line segments parallel to the axes. Actually, we insist that $\xi$ is of minimal length (as direct as possible from $z$ to $z + h$, to ensure so that we can use the continuity of $f$ to bound part of the integral, as follows. Since $f$ is continuous, write $f(w) = f(z) + \psi(w)$, where $\psi(w) \to 0$ as $w \to z$. Thus
\[
\int_\xi f(w) \, dw = f(z) \int_\xi dw + \int_\xi \psi(w) \, dw.
\]
Since $\int_\xi dw = h$ and
\[
\left| \int_\xi \psi(w) \right| \leq |h| \sup_{w \in \xi} |\psi(w)|,
\]
with the supremum tending to 0 as $h \to 0$, we see that
\[
\lim_{h \to 0} \frac{F(z + h) - F(z)}{h} = f(z),
\]
as desired.

9. As stated, the problem is false since, for instance, $f(z) = z + z^2$ has $f(0) = 0$, $f'(0) = 1$, but $f$ is not the identity map. Clearly we are meant to assume that $f$ maps $Q$ to $Q$. Let $z_0 \in Q$ denote the point at which $f(z_0) = z_0$ and $f'(z_0) = 1$. I want to use an argument similar to the Schwarz lemma but I don’t know how. Riemann mapping theorem to map $Q$ to the unit disc? But then the derivative may change...

10. The idea is to rewrite the integral as a complex integral that reduces to the desired integral when we use the parametrization $z = e^{i\theta}$. Let $\gamma$ be the
ccw. unit circle and consider the integral

\[ \frac{1}{i} \int_{\gamma} \frac{1}{z(a + \frac{z-1}{2a})} \, dz = \frac{1}{i} \int_{\gamma} \frac{2i}{2ai z + z^2 - 1} \, dz = 2 \int_{\gamma} \frac{1}{(z + i(a - \sqrt{a^2 - 1}))(z + i(a + \sqrt{a^2 - 1}))} \, dz. \]

From the left side, we see that taking \( z = e^{i\theta} \) and \( dz = ie^{i\theta} \, d\theta \) gives the desired integral. On the other hand, the right side shows that the integrand has poles at \(-i(a \pm \sqrt{a^2 - 1})\). The one corresponding to (+) clearly does not lie in the unit circle, but we claim that \(-i(a - \sqrt{a^2 - 1})\) always does for \( a > 1 \), namely that \( f(a) := a - \sqrt{a^2 - 1} < 1 \). For this it suffices to argue that \( f(1) = 1 \), and that \( f'(a) < 0 \) for \( a > 1 \), hence \( f \) is strictly decreasing for \( a > 1 \). This is easy:

\[ f'(a) = 1 - \frac{2a}{2\sqrt{a^2 - 1}} = 1 - \frac{a}{\sqrt{a^2 - 1}} < 1 - 1 = 0. \]

Now we proceed to use the residue theorem. The residue at the pole \(-i(a - \sqrt{a^2 - 1})\) is

\[ \lim_{z \to -i(a - \sqrt{a^2 - 1})} \frac{2}{z - i(a + \sqrt{a^2 - 1})} = \frac{2}{-2ai} = \frac{i}{a}, \]

so the residue theorem shows that the integral is

\[ 2\pi i \frac{i}{a} = -\frac{2\pi}{a}. \]
Chapter 6

Fall 2008

6.1 Real Analysis

1. (a) $T_a$ is clearly linear since multiplication in $\mathbb{R}$ distributes over addition. $T_a x \in \ell^\infty$ since

$$\|T_a x\|_\infty = \sup |(T_a x)_n| = \sup \left| \sum_{i=1}^{n} a_i x_i \right| \leq \sum_{i=1}^{\infty} |a_i x_i| \leq \|x\|_\infty \|a\|_1 < \infty,$$

which also shows that $\|T_a\| \leq \|a\|_1$.

(b) To prove equality, note first that if $a = 0$, then $T_a = 0$ and the result is trivial. So suppose $a \neq 0$ and let $x$ be the sequence defined by

$$x_n = \begin{cases} 
\frac{a_n}{|a_n|} & \text{when } a_n \neq 0; \\
0 & \text{when } a_n = 0.
\end{cases}$$

($x_n$ is the sign of $a_n$ or 0). Then $\|x_n\|_\infty = 1$ since $a \neq 0$, and

$$\sup \left| \sum_{i=1}^{n} a_i x_i \right| = \sup \left| \sum_{i=1}^{n} |a_i| \right| = \|a\|_1 = \|a\|_1 \|x_n\|_\infty.$$

2. (a) Suppose $f \in L^2$. Let $E$ be those points $x \in X$ for which $|f(x)| \leq 1$. Then

$$\int |f| = \int_E |f| + \int_{X-E} |f| \leq \mu(E) + \int_{X-E} |f|^2 \leq \mu(E) + \|f\|^2_2 < \infty,$$

so $f \in L^1$. 

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Since \(|f|\) is measurable and non-negative, there is a sequence of simple functions \(\{\varphi_n\}\) such that \(\varphi_n \nearrow |f|\). Thus the monotone convergence theorem guarantees that \(\int \varphi_n \to \int |f|\), and likewise \(\int \varphi_n^2 \to \int |f|^2\). If we can prove that

\[
\|\varphi\|_1 \leq \sqrt{\mu(X)}\|\varphi\|_2
\]

holds for all non-negative simple functions \(\varphi\), then we get the inequality for all \(f \in L^2\) since

\[
\|\varphi\|_1 \leq \sqrt{\mu(X)}\|\varphi\|_2 \leq \sqrt{\mu(X)}\|f\|_2
\]

by monotonicity, and the inequality is preserved upon taking the limit of the left side.

So let \(\varphi\) be a simple function, with canonical form \(\varphi_n = \sum_{k=1}^{N} a_k \chi_{E_k}\), where the \(E_k\) are disjoint finite measurable sets and the \(a_k\) are distinct positive real numbers. Then

\[
\left(\int \varphi\right)^2 = \left(\sum_{k=1}^{N} a_k \mu(E_k)\right)^2 = \sum_{k=1}^{N} a_k^2 \mu(E_k)^2 + \sum_{1 \leq j < k \leq N} a_j a_k \mu(E_j) \mu(E_k). \quad (*)
\]

On the other hand,

\[
\left(\sum_{k=1}^{N} \mu(E_k)\right) \int \varphi^2 = \left(\sum_{k=1}^{N} \mu(E_k)\right) \left(\sum_{k=1}^{N} a_k^2 \mu(E_k)\right)
= \sum_{k=1}^{N} a_k^2 \mu(E_k)^2 + \sum_{1 \leq j < k \leq N} (a_j^2 + a_k^2) \mu(E_j) \mu(E_k). \quad (**)
\]

Since \(a_j a_k \leq a_j^2 + a_k^2\) for non-negative real numbers \(a_j, a_k\), we see that \((*) \leq (**). Thus

\[
\|\varphi_n\|_1 = \int \varphi_n \leq \sqrt{\sum_{k=1}^{N} \mu(E_k) \cdot \|\varphi_n\|_2} \leq \sqrt{\mu(X)}\|\varphi_n\|_2,
\]

as desired.

To see that \(||i|| \geq \sqrt{\mu(X)}\), simply take \(f \equiv 1\).

(b) Suppose for contradiction that there is a sequence \(\{E_n\}\) of measurable sets such that \(\mu(E_n) \to 0\). Taking a subsequence if necessary, we may assume \(\mu(E_n) \leq \frac{1}{2^n}\). Then define a sequence of simple functions \(\{\varphi_n\}\) by

\(\varphi_n = \frac{1}{\sqrt{\mu(E_n)}} \chi_{E_n}\). Then

\[
\left\|\sum_{n=1}^{N} \varphi_n\right\|_1 \leq \sum_{n=1}^{N} \int \varphi_n = \sum_{n=1}^{N} \sqrt{\mu(E_n)} \leq \sum_{n=1}^{\infty} \sqrt{\mu(E_n)} \leq \sum_{n=1}^{\infty} \frac{1}{2n} \leq 1.
\]

Taking the limit of the left side, we see that \(f := \sum_{n=1}^{\infty} \varphi_n\) is well-defined and in \(L^1\). But

\[
\|f\|_2^2 \geq \sum_{n=1}^{N} \int \varphi_n^2 = \sum_{n=1}^{N} 1 = N
\]
for each $N$, whence $f \notin L^2$, contradicting the assumption.

3. For only if, suppose $f = zg$ for some $z \in \mathbb{C}^*$. If $h \in H$ such that $\langle g, h \rangle = 0$, then 
\[ \langle f, h \rangle = \langle zg, h \rangle = z \langle g, h \rangle = 0, \]
so there is no $h$ with the stated properties.

For if, suppose that for every $h \in H$ orthogonal to $g$, $\langle f, h \rangle \neq 1$. Then in fact $\langle f, h \rangle = 0$ is necessary since otherwise we could scale $h$ appropriately. Let $S$ be the subspace spanned by $g$, which is closed, and consider $H = S \oplus S^\perp$. Choose $\lambda$ such that $f = \lambda g + h$, where $h \in S^\perp$. Then $h \perp g$ and therefore $h \perp f$ imply the equalities
\[ \lambda \langle g, g \rangle = \langle f, g \rangle = 1, \]
whence $\|f\| = |\lambda| \|g\|$. But the Pythagorean theorem gives
\[ \|f\|^2 = |\lambda|^2 \|g\|^2 + \|h\|^2, \]
whence $h = 0$, so $f = \lambda g$ as desired.

4. Define $g(x, y) = f(x)\chi_{[y,1] \times [0,1]}(x, y)$, which is non-negative and measurable since $f(x)$ is measurable on $[0, 1] \times [0, 1]$ (see Stein 85). Note that $g$ is really just
\[ g(x, y) = \begin{cases} f(x) & \text{if } x \geq y; \\ 0 & \text{otherwise.} \end{cases} \]
By Fubini’s theorem, we obtain
\[ \int_{[0,1]} \left( \int_{[0,1]} f(x)\chi_{[y,1] \times [0,1]}(x, y) \, dy \right) \, dx = \int_{[0,1]} \left( \int_{[0,1]} f(x)\chi_{[y,1] \times [0,1]}(x, y) \, dx \right) \, dy, \]
in the extended sense, where the left side yields
\[ \int_{[0,1]} x f(x) \, dx \]
and the right side yields
\[ \int_{[0,1]} \left( \int_{[0,y]} f(x) \, dx \right) \, dy. \]
This proves the claim.

5. Skip.
Chapter 7

Spring 2008

7.1 Real Analysis

1. Set \( f_n := f\chi_{E_n} \), where 
\[ E_n := \{ x \in X : f(x) \leq n \} \].
Then \( f_n \nearrow f \), so \( \int f_n \to \int f \). Thus given \( \epsilon > 0 \), we can choose \( N > 0 \) such that \( \int f - \int f_n < \epsilon/2 \) whenever \( n \geq N \). Now if we choose \( \delta > 0 \) such that \( \delta < \epsilon/2N \), then for any measurable \( E \subseteq X \) with \( \mu(E) < \delta \), we compute
\[
\int_E f = \int_E (f - f_n) + \int_E f_n \leq \int_E (f - f_n) + n \cdot \mu(E) < \frac{\epsilon}{2} + \frac{n\epsilon}{2N} \leq \epsilon,
\]
which proves the result.

2. Since \( K \) is continuous on a compact subset of \( \mathbb{R}^2 \), \( K \) is bounded, namely \( K \in L^\infty([0, 1] \times [0, 1]) \), with \( |K| \) having supremum \( \| K \|_\infty \). Then
\[
|(Tf)(x)| = \left| \int_{[0, 1]} K(x, y)f(y)\,dy \right| \leq \int_{[0, 1]} |K(x, y)||f(y)|\,dy \leq \| K \|_\infty \| f \|_1,
\]
so \( Tf \) is in fact uniformly bounded, hence square integrable on \([0, 1] \), a set with finite measure. Since in fact \([0, 1] \) has measure 1, the above inequality implies that
\[
\| Tf \|_1 := \int_{[0, 1]} |Tf| \leq \int_{[0, 1]} \|K\|_\infty \|f\|_1 \leq \|K\|_\infty \|f\|_1,
\]
whence \( \| T \| \leq \| K \|_\infty \).
3. We can write $T = \ker T \oplus (\ker T)^\perp$, where $T|_{(\ker T)^\perp}$ gives a bijection with $\text{im}(T)$. Then $T|_{(\ker T)^\perp}$ is bounded since linear transformations between finite-dimensional Hilbert spaces are bounded, whence $T$ has the same bound.

Bounded linear transformations map closed sets to closed sets since if $x_k \to x$, then $Tx_k \to Tx$. This holds because

$$
\|Tx - Tx_k\| = \|T(x - x_k)\| \leq \|T\|\|x - x_k\|.
$$

Now, every bounded closed set of a finite-dimensional normed space is compact. For this it suffices to show that every bounded sequence in a finite-dimensional vector space has a convergent subsequence. Choosing a normal basis $\{e_1, \ldots, e_n\}$, we have $\|a_1 e_1 + \cdots + a_n e_n\| \leq |a_1| + \cdots + |a_n|$, so convergence is analogous to convergence in $\mathbb{C}^n$, which is guaranteed by Bolzano-Weierstrass.

Then if $T$ is bounded and $\{x_k\}$ is bounded, $\{Tx_k\}$ is bounded, hence has a convergent subsequence, so $T$ is compact.