Quiz #4 Solutions

(1) Compute $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$. (3 points)

Possible solution:

\[
\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{vmatrix} = 0
\]

Possible solution:

\[
\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}
\]

\[
= (45 - 48) - 2(36 - 42) + 3(32 - 35)
\]

\[
= -3 + 12 - 9 = 0
\]

Possible solution: The columns of the matrix are dependent since

\[
\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix},
\]

so the matrix is not invertible and hence its determinant is 0.

(2) Let $A$ and $B$ be $5 \times 5$ matrices, with $\det A = -2$ and $\det B = 3$. Use properties of determinants to compute each of the following: (2 points)

Solution:

(a) $\det BA = (\det B)(\det A) = 3(-2) = -6$

(b) $\det 2A = 2^5 \det A = -2^6 = -64$
(3) Compute the area of the parallelogram whose vertices are \( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \). (2 points)

Solution: Let \( S \) be the unit square with vertices \( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) and let \( T: \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear transformation defined by \( T(\bar{x}) = A\bar{x} \), where \( A = \begin{bmatrix} 4 & -1 \\ 3 & 2 \end{bmatrix} \). Then \( T(S) \) is the parallelogram we are interested in and its area is given by

\[
\text{area of } T(S) = (\text{area of } S) |\det A| = 1 \cdot \left| \begin{bmatrix} 4 & -1 \\ 3 & 2 \end{bmatrix} \right| = |8 - (-3)| = 11.
\]

(4) Bonus problem: Let \( a \) and \( b \) be positive numbers. Compute the area of the region in \( \mathbb{R}^2 \) bounded by the ellipse whose equation is

\[
\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1.
\]

(1 bonus point)

Solution: Let \( S \) be the region in \( \mathbb{R}^2 \) bounded by the unit circle, which has equation \( u_1^2 + u_2^2 = 1 \). Let \( T: \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear transformation defined by \( T(\bar{u}) = A\bar{u} \), where \( A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \). Then \( T(S) \) is the ellipse we are interested in. (To see this, note that the image of a vector \( \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \) on the unit circle is the vector \( T\left( \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) = \begin{bmatrix} au_1 \\ bu_2 \end{bmatrix} \), which satisfies the equation for the ellipse:

\[
\frac{(au_1)^2}{a^2} + \frac{(bu_2)^2}{b^2} = u_1^2 + u_2^2 = 1.
\]

Thus \( T \) maps the unit circle to the ellipse. Since \( T \) is linear, \( T \) must map the interior of the unit circle to the interior of the ellipse.) Now by the area formula,

\[
\text{area of } T(S) = (\text{area of } S) |\det A| = \pi \cdot 1^2 \cdot \left| \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \right| = \pi ab.
\]