Representations for the Conductivity Functions of Multicomponent Composites

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Abstract

The effective conductivity σ^* of a multicomponent composite material is considered. Integral representations for σ^* treated as a holomorphic function on a polydisk with values in a half-plane are analyzed. A representation for σ^* is introduced which is symmetric in the component conductivities and for which the moments of the positive measure in the integral are directly related to the coefficients in a perturbation expansion of σ^* around a homogeneous medium. This second feature, which is important for obtaining bounds on σ^*, was previously available only in the two-component case. In addition, a bound valid for any holomorphic function of the above type is proven.

1. Introduction

The effective conductivity of a composite of n isotropic constituent materials depends both on the conductivities $\sigma_1, \sigma_2, \cdots, \sigma_n$ of the components and on their geometrical configuration. This is clearly evident if we consider the limit in which one of the components becomes infinitely conducting. Then the effective conductivity becomes infinite or not according to whether there is a continuous path of that component across the composite. In this article we seek integral representations for the effective conductivity which separate the analytic dependence on $\sigma_1, \sigma_2, \cdots, \sigma_n$ from the dependence on the geometric configuration.

These representations involve a kernel containing the σ_i which is integrated with respect to a measure containing the information on the geometry. One of the principal uses of the integral formulas has been to derive bounds on the effective conductivity σ^* assuming information about the geometry of the medium, which puts constraints on the measure.

We shall consider two types of representations, asymmetric and symmetric. The first type is based on the fact that σ^* is a homogeneous function of the σ_i, so that it can be considered as an analytic function of only $n-1$ variables, eliminating, say, σ_q. Representations based on this observation were considered for two-component media by Bergman [2] and Fuchs [8], and for multicomponent media by Golden and Papanicolaou [10]–[12]. The measures in these

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representations are positive, which plays an important role in obtaining bounds on σ^*. Here we derive relations between the measures associated with elimination of σ_p and σ_q, $p \neq q$.

The second type of representation formula we consider is symmetric in the σ_i, it treats them all on an equal footing. Such analytic formulas were considered by Dell'Antonio and Nesi [7], although their representation incorporates measures which are not necessarily positive. In this paper we introduce a new symmetric representation which has a positive measure and solves the following problem. In obtaining bounds on σ^*, geometric information is most conveniently introduced into the representation via the moments of the measure. In the most common asymmetric formula for two-component media (see [2], [8], [9], [16]) the moments of the measure are directly related to the coefficients of a natural perturbation expansion of σ^* around a homogeneous medium ($\sigma_1 = \sigma_2$), which contains the geometrical information. In the multicomponent version of this procedure (see [10]–[12]), the relation between the moments of the measure and the coefficients of the perturbation expansion is quite complicated. The representation we introduce here has the important feature that the moments of the measure are directly related to the coefficients of a perturbation expansion.

In an appendix we give a proof of an elementary bound on functions analytic in the polydisk with positive real part there. The extremal measures which yield the bound also provide some optimal bounds on σ^* which were conjectured by Golden [10]–[12] in the multicomponent case.

For simplicity we assume that the electric and current fields and the geometry of the composite are periodic with a primitive cell Q of unit volume. We could equally well treat the random case: Golden and Papanicolaou [9] have shown how the analysis can be extended to include ensembles of materials.

Locally, conduction is governed by the equations

$$J(x) = \sigma(x)E(x), \tag{1.1}$$

$$\nabla \cdot J(x) = 0, \qquad \nabla \times E(x) = 0, \tag{1.2}$$

$$\sigma(x) = \sum_{i=1}^{n} \sigma_i \chi_i(x), \tag{1.3}$$

where $J(x)$, $E(x)$ and $\sigma(x)$ are the Q-periodic current field, electric field and conductivity, and the $\chi_i(x)$ are the characteristic functions

$$\chi_i(x) = \begin{cases} 1 & \text{in component } i, \\ 0 & \text{elsewhere.} \end{cases} \tag{1.4}$$

On a macroscopic scale, the average current field and average electric field,

$$j^* = \int_Q J(x)\,dx, \qquad e^* = \int_Q E(x)\,dx \tag{1.5}$$

are linearly related,

$$j^* = \boldsymbol{\sigma}^* e^*, \tag{1.6}$$

and the tensor $\boldsymbol{\sigma}^*$ that relates them is called the effective conductivity tensor. Rather than studying the analytic behavior of the full tensor $\boldsymbol{\sigma}^*$ we shall focus our attention on the diagonal element

$$\sigma^* = \sigma^*(e^*) = e^* \cdot \boldsymbol{\sigma}^* e^*, \tag{1.7}$$

where e^* is a fixed real unit vector.

To derive the standard formula for σ^*, which we shall call the effective conductivity, let us introduce the polarization field

$$P(x) = [\sigma(x) - \sigma_0]E(x) = J(x) - \sigma_0 E(x); \tag{1.8}$$

where σ_0 is the conductivity of an isotropic homogeneous reference medium, which may be chosen freely. The polarization field satisfies

$$(\Gamma - s)P(x) = \sigma_0 e^*, \tag{1.9}$$

where

$$s(x) = \sigma_0(\sigma_0 - \sigma(x))^{-1} = \sum_{i=1}^{n} s_i \chi_i(x), \tag{1.10}$$

$$s_i = \sigma_0(\sigma_0 - \sigma_i)^{-1}, \tag{1.11}$$

and Γ is the projection operator (in $L^2(Q)$) onto the subspace $\mathscr{E}$ of curl-free, mean zero Q-periodic fields,

$$\mathscr{E} = \left\{ E'(x) \,|\, \nabla \times E'(x) = 0, \int_Q E'(x)\,dx = 0 \right\}. \tag{1.12}$$

In Fourier space, Γ is a local projection operator with Fourier components

$$\hat{\Gamma}(k)_{ij} = \begin{cases} k_i k_j / |k|^2 & \text{when } |k| \neq 0, \\ 0 & \text{when } |k| = 0, \end{cases} \tag{1.13}$$

and consequently we have

$$\Gamma J(x) = 0, \qquad \Gamma E(x) = E(x) - e^*. \tag{1.14}$$

From these relations (1.9) is easily established.

Now (1.8) and (1.9) imply

$$\sigma^* e^* - \sigma_0 e^* = \int_Q P(x)\,dx = \int_Q (\Gamma - s)^{-1}\sigma_0 e^*. \tag{1.15}$$

This gives the formula

$$\sigma^* = \sigma_0 + \sigma_0 \int_Q e^* \cdot (\Gamma - s)^{-1} e^*\,dx \tag{1.16}$$

for the effective conductivity σ^*. The expression is not as useful as it may seem because the operator $\Gamma - s$ is difficult to invert: Γ is local only in Fourier space whereas s is local only in real space.

Integral representations (see [2], [7], [8], [11]) for the effective conductivity function $\sigma^*(\sigma_1, \sigma_2, \cdots, \sigma_n)$ follow from its analytic properties. Dell-Antonio, Figari, and Orlandi [6] noted that the function is analytic on the domain

$$\Delta = \bigcup_{\alpha \in [0, 2\pi)} H_\alpha^n, \tag{1.17}$$

where H_α^n is the n-fold product of half-planes,

$$H_\alpha^n = \left\{\mathcal{Re}\left(e^{-i\alpha}\sigma_1\right) > 0\right\} \times \left\{\mathcal{Re}\left(e^{-i\alpha}\sigma_2\right) > 0\right\} \times \cdots \times \left\{\mathcal{Re}\left(e^{-i\alpha}\sigma_n\right) > 0\right\}. \tag{1.18}$$

On this domain Δ, the function σ^* is *homogeneous*

$$\sigma^*(\lambda\sigma_1, \lambda\sigma_2, \cdots, \lambda\sigma_n) = \lambda\sigma^*(\sigma_1, \sigma_2, \cdots, \sigma_n) \tag{1.19}$$

for all $\lambda \in \mathbb{C}$, satisfies the *normalization*

$$\sigma^*(1, 1, \cdots, 1) = 1, \tag{1.20}$$

and the *energy dissipation property*

$$\sigma^* \in H_0 \quad \text{whenever} \quad (\sigma_1, \sigma_2, \cdots, \sigma_n) \in H_0^n. \tag{1.21}$$

All these properties are implied by the formula (1.16) for the effective conductivity.

By taking $\lambda = e^{-i\alpha}$ we deduce from (1.19) and (1.21) that

$$\sigma^* \in H_\alpha \quad \text{whenever} \quad (\sigma_1, \sigma_2, \cdots, \sigma_n) \in H_\alpha^n. \tag{1.22}$$

Consequently, if $W(\alpha, \beta)$ denotes the wedge

$$W(\alpha, \beta) = H_\alpha \cap H_\beta \tag{1.23}$$

in the complex plane, then (1.22) implies the *wedge bounds*

$$\sigma^* \in W(\alpha, \beta) \quad \text{whenever} \quad \sigma_i \in W(\alpha, \beta) \quad \text{for all } i. \tag{1.24}$$

The representation formulas we develop apply to any function, analytic on Δ, which satisfies the three properties (1.19), (1.20) and (1.21). Such functions will be called conductivity functions.

For two-component composites Bergman [2] deduced the integral representation

$$\sigma^* = \sigma_2 - \sigma_2 \int_0^1 \frac{d\mu(z)}{s_1 - z} \tag{1.25}$$

for the conductivity function, where μ is a positive measure depending on $\chi_1(x)$ and

$$s_1 = \sigma_2/(\sigma_2 - \sigma_1). \tag{1.26}$$

A rigorous proof of this representation was given by Golden and Papanicolaou [9]. The representation clearly separates the analytic dependence of σ^* on σ_1 and σ_2 from the dependence on the geometry: the measure μ characterizes the relevant features of the composite geometry and does not depend on the values of σ_1 and σ_2. Similar integral representations were obtained independently by Lysne [17] and Fuchs [8] based on specific models of the composite; see also Korringa [16].

Bergman's integral representation (1.25) has been successfully used to bound the effective conductivity σ^* given limited information about the composite geometry. When $|s_1| > 1$, the denominator in (1.25) can be expanded and we obtain the series

$$\sigma^* = \sigma_2 - \sigma_2 \sum_{i=0}^{\infty} \mu^{(i)}/s_1^{i+1} \tag{1.27}$$

for the effective conductivity in terms of the moments

$$\mu^{(i)} = \int_0^1 z^i\,d\mu(z) \tag{1.28}$$

of the measure. This series is an expansion about a homogeneous medium $\sigma_1 = \sigma_2$ ($s = \infty$), and the representation (1.25) provides the analytic continuation of this series to the full domain of analyticity of σ^*. Alternatively, by choosing $\sigma_0 = \sigma_2$

and expanding (1.16) in powers of $1/s_1$ we deduce that

$$\mu^{(i)} = \int_Q e^* \cdot (\chi_1 \Gamma)^i \chi_1 e^*. \tag{1.29}$$

In particular we have

$$\mu^{(0)} = p_1, \tag{1.30}$$

where p_1 is the volume fraction occupied by component 1. In general, $\mu^{(i)}$ depends on the correlation function that gives the probability that a configuration of $i+1$ points lands with all points in component 1 when placed randomly in the composite; see [5]. For isotropic materials the expression for $\mu^{(1)}$ reduces to

$$\mu^{(1)} = p_1 p_2 / d, \tag{1.31}$$

where d is the dimensionality of the composite.

The relations (1.30) and (1.31) can be regarded as constraints on the measure μ. The measure must also satisfy the constraint

$$\int_0^1 \frac{d\mu(z)}{1-z} \leqq 1, \tag{1.32}$$

which follows because the wedge bounds imply that $\sigma^* \geqq 0$ when $\sigma_1, \sigma_2 \geqq 0$. By finding the extreme values of σ^*, given by (1.25), as the measure μ is varied over all positive measures that satisfy (1.30), (1.31), and (1.32), Bergman [2] obtained the Hashin-Shtrikman bounds

$$\sigma_1 + \frac{p_2}{1/(\sigma_2 - \sigma_1) + p_1/d\sigma_1} \leqq \sigma^* \leqq \sigma_2 + \frac{p_1}{1/(\sigma_1 - \sigma_2) + p_2/d\sigma_2} \tag{1.33}$$

on the effective conductivity of isotropic composites with $\sigma_2 \geqq \sigma_1$. These bounds were first derived from variational principles by Hashin and Shtrikman [14]. They found that the upper and lower limits correspond to the conductivity of assemblages of coated spheres (with sizes ranging to the infinitesimal) packed to fill all space.

Subsequently, in independent work, Milton [20] and Bergman [3] used the integral representation formula (1.25) to extend the Hashin-Shtrikman bounds, and other bounds, to complex ratios of σ_1 and σ_2. Complex conductivities are needed to describe the response of the composite to oscillating applied fields in the regime where the wavelength and attenuation lengths are much larger than the composite inhomogeneities.

The representation formula (1.25) has also been useful in estimating the volume fractions of the components given measurements of the complex conductivity at various frequencies (see [18]) and for modeling the frequency-dependent

response of metallic particles in a ceramic matrix (see [8]) and of brine-saturated porous rocks (see [13], [23]).

These successes motivate us to seek integral representations for the conductivity function of multicomponent composites. Considerable progress has already been made. Golden and Papanicolaou [10], [11] developed an integral representation formula based on the observation, implied by (1.22), that

$$\mathscr{I}m(\sigma^*/\sigma_q) \geqq 0 \quad \text{when} \quad \mathscr{I}m(\sigma_i/\sigma_q) \geqq 0 \quad \text{for all} \quad i, \tag{1.34}$$

where σ_q, the reference variable, represents one of the conductivities $\sigma_1, \sigma_2, \cdots, \sigma_n$. Their asymmetric representation incorporates a positive measure μ which satisfies certain Fourier constraints. A modified version of this representation is analyzed in Section 3. Some additional constraints on the measure are derived in Section 4. In Section 5 we develop an integral representation for multicomponent media such that the moments or Fourier coefficients of the measure are related to the coefficients in a perturbation expansion of σ^* of a nearly homogeneous composite.

2. A Representation Formula for Analytic Functions of Several Complex Variables

We begin by giving an integral representation formula for a function of m complex variables $z_1, z_2, \cdots, z_m$ when that function is analytic within the unit polydisk

$$D^m = \{|z_1| < 1\} \times \{|z_2| < 1\} \times \cdots \times \{|z_m| < 1\} \tag{2.1}$$

and takes values in a half-plane. Given points $z = \{z_1, z_2, \cdots, z_m\}$ and $w = \{w_1, w_2, \cdots, w_m\}$ let us introduce the Szëgo kernel

$$S(z, w) = \prod_{k=1}^{m} \frac{1}{1 - z_k \overline{w}_k} = \sum_{n_1, n_2, \cdots, n_m \geqq 0} \overline{w}_1^{n_1} \cdots \overline{w}_m^{n_m} z_1^{n_1} \cdots z_m^{n_m}, \tag{2.2}$$

on $D^m \times T^m$, where

$$T^m = \{|w_1| = 1\} \times \{|w_2| = 1\} \times \cdots \times \{|w_m| = 1\} \tag{2.3}$$

is the distinguished boundary of D^m and is denoted as the torus. The torus is conveniently parameterized by the angular variables $\theta_1, \theta_2, \cdots, \theta_m$ defined via the relation

$$w_j = \exp(i\theta_j) \quad \text{with} \quad 0 \leqq \theta_j < 2\pi. \tag{2.4}$$

We then have the following well-established theorem; see [15].

THEOREM 1. *A function $f(z)$ is analytic for $z = \{z_1, z_2, \cdots, z_m\} \in D^m$ and has non-negative real part there,*

$$\mathcal{R}e\, f(z) \geq 0 \quad \text{for all} \quad z \in D^m, \tag{2.5}$$

if and only if it has the integral representation

$$f(z) = i\,\mathcal{I}m\, f(0) + \int_{T^m} (2S(z,w) - 1)\, d\mu(\theta_1, \theta_2, \cdots, \theta_m), \tag{2.6}$$

where the measure

$$\begin{aligned} d\mu(\theta_1, \theta_2, \cdots, \theta_m) = \lim_{r \to 1^-} (2\pi)^{-m} \mathcal{R}e\big[f(r\exp\{i\theta_1\}, r\exp\{i\theta_2\}, \\ \cdots, r\exp\{i\theta_m\})\big]\, d\theta_1 \cdots d\theta_m \end{aligned} \tag{2.7}$$

is positive and satisfies the Fourier condition

$$\begin{aligned} a_{k_1, k_2, \cdots, k_n} &\equiv \int_{T^m} \exp\{i(k_1\theta_1 + k_2\theta_2 + \cdots + k_m\theta_m)\}\, d\mu(\theta_1, \theta_2, \cdots, \theta_m) \\ &= 0 \quad \text{unless} \quad (k_1, k_2, \cdots, k_m) \in \mathbb{Z}_+^m \cup \mathbb{Z}_-^m, \end{aligned} \tag{2.8}$$

where $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$ and $\mathbb{Z}_- = -\mathbb{Z}_+$.

An examination of the proof of this theorem (see [15]) shows that the expression (2.7) for the measure can be replaced by

$$\begin{aligned} d\mu(\theta_1, \theta_2, \cdots, \theta_m) = \lim_{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n \to 0^+} (2\pi)^{-m} \mathcal{R}e\big[f(\exp\{i\theta_1 - \varepsilon_i\}, \\ \cdots, \exp\{i\theta_m - \varepsilon_m\})\big]\, d\theta_1 \cdots d\theta_m, \end{aligned} \tag{2.9}$$

i.e., the measure is independent of the way the positive infinitesimals $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m$ are taken to zero: there is no need to impose the restriction that $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_m$.

The non-zero Fourier components of the measure μ are directly related to the series expansion of the function $f(z)$ around the origin $z = 0$ of the polydisk. By substituting (2.2) into (2.6) we have

$$\begin{aligned} f(z) &= i\,\mathcal{I}m\, f(0) \\ &+ 2\Bigg(\sum_{n_1, n_2, \cdots, n_m \geq 0} a_{n_1, n_2, \cdots, n_m} z_1^{n_1} z_2^{n_2} \cdots z_m^{n_m}\Bigg) - a_{0,0,\cdots,0}. \end{aligned} \tag{2.10}$$

Thus the function $f(z)$ has the series expansion

$$f(z) = \sum_{n_1, n_2, \cdots, n_m \geq 0} c_{n_1, n_2, \cdots, n_m} z_1^{n_1} z_2^{n_2} \cdots z_m^{n_m}, \tag{2.11}$$

where the coefficients $c_{n_1, n_2, \cdots, n_m}$ are given by

$$c_{n_1, n_2, \cdots, n_m} = \begin{cases} a_{0,0,\cdots,0} + i\,\mathcal{I}m\, f(0) & \text{if } n_1 = \cdots = n_m = 0, \\ 2a_{n_1, n_2, \cdots, n_m} & \text{otherwise.} \end{cases} \tag{2.12}$$

3. An Asymmetric Representation Formula for the Conductivity Function

The analysis of this section is based on the work of Golden and Papanicolaou [10], [11]. We obtain a representation formula for the n-variable conductivity function in terms of a measure on the $(n-1)$ torus T^{n-1}. The representation is asymmetric because one of the variables, the reference variable, plays a special role in the representation.

Let us choose σ_q for some $q \in \{1, 2, \cdots, n\}$ as our reference variable. Let us also define the conductivity ratios

$$h_j^{(q)} = \sigma_j / \sigma_q \tag{3.1}$$

for $j = 1$ up to $j = m$, excluding $j = q$. We assume that the dummy index j in any variable $x_j^{(q)}$, such as $h_j^{(q)}$, runs over the set $\{1, 2, \cdots, q-1, q+1, \cdots, n\}$ skipping $j = q$.

The next step is to notice that we can eliminate one of the variables from the conductivity function because the function is homogeneous of degree one (see [2]). Specifically the homogeneity relation (1.19) with $\lambda = 1/\sigma_q$ implies

$$\sigma^*(\sigma_1, \sigma_2, \cdots, \sigma_m) = \sigma_q \sigma^*\big(h_1^{(q)}, \cdots, h_{q-1}^{(q)}, 1, h_{q+1}^{(q)}, \cdots, h_m^{(q)}\big). \tag{3.2}$$

Therefore we should focus on finding a suitable representation formula for the

$(n-1)$-variable function

$$m_q(h^{(q)}) = m_q\left(h_1^{(q)},\cdots,h_{q-1}^{(q)},h_{q+1}^{(q)},\cdots,h_m^{(q)}\right) \tag{3.3}$$
$$= \sigma^*\left(h_1^{(q)},\cdots,h_{q-1}^{(q)},1,h_{q+1}^{(q)},\cdots,h_m^{(q)}\right),$$

where $h^{(q)}$ denotes the point $\{h_1^{(q)},\cdots,h_{q-1}^{(q)},h_{q+1}^{(q)},\cdots,h_m^{(q)}\}$ in complex Euclidean space. From (1.22) we have

$$\mathscr{I}m\, m_q(h^{(q)}) \geqq 0 \tag{3.4}$$

on the domain

$$U^{n-1} = \left\{\mathscr{I}m\, h_j^{(q)} > 0\right\} \times \left\{\mathscr{I}m\, h_2^{(q)} > 0\right\} \times \cdots \times \left\{\mathscr{I}m\, h_m^{(q)} > 0\right\}. \tag{3.5}$$

To apply the representation formula of Section 2 we need a one-to-one analytic mapping between the upper half-plane U and the unit disk D. There are many fractional linear transformations which do this. Let us introduce the variables

$$z_j^{(q)} = \left(i - h_j^{(q)}\right)/\left(i + h_j^{(q)}\right) \tag{3.6}$$

for all $j \neq q$, in terms of which

$$h_j^{(q)} = i\left(1 - z_j^{(q)}\right)/\left(1 + z_j^{(q)}\right). \tag{3.7}$$

The mapping (3.6) maps the upper half-plane U onto the unit disk D and takes the positive real axis to the unit semicircle in the upper half-plane: the points $h_j^{(q)} = 0$, 1, ∞ and i get mapped to $z_j^{(q)} = 1$, i, -1 and 0, respectively. Other choices of fractional linear transformations would suffice. For example, Golden and Papanicolaou [10], [11] use the fractional linear transformation $z_j^{(g)} = [1 - i(1 - h_j^{(q)})]/[1 + i(1 - h_j^{(q)})]$ which also maps U onto D. We have found, however, that the choice (3.6) simplifies subsequent formulae.

From (3.4) it follows that the function

$$f_q(z^{(q)}) = -im_q\left(\frac{i\left(1 - z_1^{(q)}\right)}{1 + z_1^{(q)}},\cdots,\frac{i\left(1 - z_m^{(q)}\right)}{1 + z_m^{(q)}}\right) \tag{3.8}$$

has positive real part on the unit polydisk D^{n-1}. Therefore the conductivity function $m_q(h_1^{(q)},\cdots,h_n^{(q)})$ has the representation

$$m_q\left(h_1^{(q)},\cdots,h_n^{(q)}\right) \tag{3.9}$$
$$= \mathscr{R}e\left(m_q(i,i,\cdots,i)\right) + i\int_{T^{n-1}}\left(2S(z^{(q)},w^{(q)}) - 1\right) d\mu_q\left(\theta_1^{(q)},\cdots,\theta_n^{(q)}\right),$$

where $z^{(q)}$ and $w^{(q)}$ represent the points

$$z^{(q)} = \left\{\frac{i - h_1^{(q)}}{i + h_1^{(q)}},\cdots,\frac{i - h_{q-1}^{(q)}}{i + h_{q-1}^{(q)}},\frac{i - h_{q+1}^{(q)}}{i + h_{q+1}^{(q)}},\cdots,\frac{i - h_n^{(q)}}{i + h_n^{(q)}}\right\}, \tag{3.10}$$

$$w^{(q)} = \left\{\exp\left(i\theta_1^{(q)}\right),\cdots,\exp\left(i\theta_{q-1}^{(q)}\right),\exp\left(i\theta_{q+1}^{(q)}\right),\cdots,\exp\left(i\theta_n^{(q)}\right)\right\}, \tag{3.11}$$

and $\theta_1^{(q)},\theta_2^{(q)},\cdots,\theta_{q-1}^{(q)},\theta_{q+1}^{(q)},\cdots,\theta_n^{(q)}$ represent angular variables that parameterize the torus T^{n-1} on which the positive measure

$$d\mu_q\left(\theta_1^{(q)},\cdots,\theta_{q-1}^{(q)},\theta_{q+1}^{(q)},\cdots,\theta_n^{(q)}\right)$$
$$= \lim_{\varepsilon_1^{(q)},\cdots,\varepsilon_n^{(q)}\to 0^+}(2\pi)^{-n+1}\mathscr{R}e\left[f_q\left(\exp\left(i\theta_1^{(q)} - \varepsilon_1^{(q)}\right),\right.\right. \tag{3.12}$$
$$\left.\left.\cdots,\exp\left(i\theta_n^{(q)} - \varepsilon_n^{(q)}\right)\right]\, d\theta_1^{(q)}\cdots d\theta_n^{(q)}$$

resides.

Notice that when $z_j^{(q)} = \exp(i\theta_j^{(q)} - \varepsilon_j^{(q)})$ the formula (3.7) for $h_j^{(q)}$ reduces to

$$h_j^{(q)} = \frac{-i\left[\exp\left\{i\frac{1}{2}\left(\theta_j^{(q)} + i\varepsilon_j^{(q)}\right)\right\} - \exp\left\{-i\frac{1}{2}\left(\theta_j^{(q)} + i\varepsilon_j^{(q)}\right)\right\}\right]}{\left[\exp\left\{i\frac{1}{2}\left(\theta_j^{(q)} + i\varepsilon_j^{(q)}\right)\right\} + \exp\left\{-i\frac{1}{2}\left(\theta_j^{(q)} + i\varepsilon_j^{(q)}\right)\right\}\right]} \tag{3.13}$$
$$= \tan\left(\tfrac{1}{2}\left(\theta_j^{(q)} + i\varepsilon_j^{(q)}\right)\right).$$

So from (3.12) and (3.8) we find that

$$d\mu_q\left(\theta_1^{(q)},\cdots,\theta_{q-1}^{(q)},\theta_{q+1}^{(q)},\cdots,\theta_n^{(q)}\right)$$
$$= \lim_{\varepsilon_1^{(q)},\cdots,\varepsilon_n^{(q)}\to 0^+}(2\pi)^{-n+1}\mathscr{I}m\left[m_q\left(\tan\tfrac{1}{2}\left(\theta_1^{(q)} + i\varepsilon_1^{(q)}\right),\right.\right. \tag{3.14}$$
$$\left.\left.\cdots,\tan\tfrac{1}{2}\left(\theta_n^{(q)} + i\varepsilon_n^{(q)}\right)\right)\right]\, d\theta_1^{(q)}\cdots d\theta_n^{(q)}.$$

Taking complex conjugates we obtain the alternative expression

$$d\mu_q\left(\theta_1^{(q)},\cdots,\theta_{q-1}^{(q)},\theta_{q+1}^{(q)},\cdots,\theta_n^{(q)}\right)$$

$$(3.15)\qquad = -\lim_{\varepsilon_1^{(q)},\cdots,\varepsilon_n^{(q)}\to 0^-}(2\pi)^{-n+1}\mathscr{I}m\left[m_q\left(\tan\tfrac{1}{2}\left(\theta_1^{(q)}+i\varepsilon_1^{(q)}\right),\cdots,\tan\tfrac{1}{2}\left(\theta_n^{(q)}+i\varepsilon_n^{(q)}\right)\right)\right]d\theta_1^{(q)}\cdots d\theta_n^{(q)}$$

for the measure.

Of course, the measure must satisfy the Fourier constraints

$$\int_{T^{n-1}}\exp\left\{i\left(k_1^{(q)}\theta_1^{(q)}+\cdots+k_n^{(q)}\theta_n^{(q)}\right)\right\}d\mu_q\left(\theta_1^{(q)},\cdots,\theta_n^{(q)}\right)=0$$

$$(3.16)\qquad \text{unless}\quad \left(k_1^{(q)},\cdots,k_{q-1}^{(q)},k_{q+1}^{(q)},\cdots,k_n^{(q)}\right)\in\mathbb{Z}_+^{n-1}\cup\mathbb{Z}_-^{n-1}.$$

Furthermore, since the wedge bounds (1.24) imply that $m_q(h^{(q)})$ is real and non-negative when the conductivity ratios $h_j^{(q)}$ are real and non-negative, it follows that the measure $d\mu_q(\theta_1^{(q)},\cdots,\theta_n^{(q)})$ is zero on the subregion

$$(3.17)\qquad T_+^{n-1}=\left\{0\leqq\theta_1^{(q)}<\pi\right\}\times\cdots\times\left\{0\leqq\theta_n^{(q)}<\pi\right\}$$

of the torus T^{n-1}. In some sense the subregion T_+^{n-1} is like a multidimensional square "patch" on the torus T^{n-1}. The measure must vanish on this patch.

4. Additional Constraints on the Measure in the Asymmetric Representation Formula

The measure μ_q is clearly positive, satisfies the Fourier constraints (3.16), and is zero on the region T_+^{n-1}. There are, however, additional constraints which arise because we have not accounted for all the analytic properties of the conductivity function. Specifically we have not accounted for the fact that

$$(4.1)\qquad \mathscr{I}m\, m_p\left(h^{(p)}\right)>0$$

on the domain

$$(4.2)\qquad U_p^{n-1}=\left\{\mathscr{I}m\, h_1^{(p)}>0\right\}\times\cdots\times\left\{\mathscr{I}m\, h_m^{(p)}>0\right\}$$

for $p=1$ up to $p=n$ and not just for $p=q$.

Equivalently we have not accounted for the fact that the family of measures μ_p are positive and satisfy the Fourier constraints for $p=1$ up to $p=n$ and not just for $p=q$. Let us suppose that $p\neq q$ and let us try to express μ_p in terms of μ_q.

The first step is to obtain a relation between the angular parameters on the two tori. The identity

$$(4.3)\qquad \sigma_j/\sigma_q=\left(\sigma_j/\sigma_p\right)/\left(\sigma_q/\sigma_p\right)$$

implies

$$(4.4)\qquad h_j^{(q)}=\begin{cases}h_j^{(p)}/h_q^{(p)} & \text{when } j\neq p,\\ 1/h_q^{(p)} & \text{when } j=p.\end{cases}$$

Hence we have

$$(4.5)\qquad \begin{aligned}\tan\tfrac{1}{2}\theta_j^{(q)}&=\tan\tfrac{1}{2}\theta_j^{(p)}/\tan\tfrac{1}{2}\theta_q^{(p)},\quad j\neq p\\ \theta_p^{(q)}&=\pi-\theta_q^{(p)}.\end{aligned}$$

This non-linear map defines a correspondence between points on the torus on which μ_q resides and points on the torus on which μ_p resides.

Notice that when $\theta_q^{(p)}=0$ and $\theta_j^{(p)}\neq 0$ for all $j\neq q$ we have $\theta^{(q)}=(\pi,\pi,\cdots,\pi)$ irrespective of the values of $\theta_j^{(p)}$, i.e., many points get mapped to a single point. Similarly when $\theta_q^{(p)}=\pi$ and $\theta_j^{(p)}\neq\pi$ for all $j\neq q$ we have $\theta^{(q)}=(0,0,\cdots,0)$ irrespective of the values of $\theta_j^{(p)}$. To avoid complications resulting from this non-uniqueness, which would prolong the analysis, let us only consider "suitable" conductivity functions. A "suitable" conductivity function is one for which the distribution functions

$$g_i\left(\theta_1^{(i)},\cdots,\theta_n^{(i)}\right)$$

$$(4.6)\qquad =\lim_{\delta_1^{(i)},\cdots,\delta_n^{(i)}\to 0^+}\mathscr{I}m\left[m_i\left(\left(\tan\tfrac{1}{2}\theta_1^{(i)}\right)+i\delta_1^{(i)},\cdots,\left(\tan\tfrac{1}{2}\theta_n^{(i)}\right)+i\delta_n^{(i)}\right)\right]$$

exist for all i, and for which there is negligible contribution to the measure in the vicinity of points where $\theta_j^{(i)}=\pi$ or 0 for some j, specifically for which

$$(4.7)\qquad \lim_{\varepsilon\to 0}\int_{S_{\varepsilon,j}}d\mu_i\left(\theta_1^{(i)},\cdots,\theta_n^{(i)}\right)=0,$$

$$(4.8)\qquad \lim_{\varepsilon\to 0}\int_{S'_{\varepsilon,j}}d\mu_i\left(\theta_1^{(i)},\cdots,\theta_n^{(i)}\right)=0,$$

for all i and all j, where $S_{\varepsilon,j}$ and $S'_{\varepsilon,j}$ are the strips

$$(4.9)\qquad \begin{aligned} S_{\varepsilon,j} &= \{0 \leqq \theta_1^{(i)} < 2\pi\} \times \cdots \times \{\pi - \varepsilon < \theta_j^{(i)} < \pi + \varepsilon\} \\ &\quad \times \cdots \times \{0 \leqq \theta_n^{(i)} < 2\pi\}, \\ S'_{\varepsilon,j} &= \{0 \leqq \theta_1^{(i)} < 2\pi\} \times \cdots \times \{-\varepsilon < \theta_j^{(i)} < \varepsilon\} \\ &\quad \times \cdots \times \{0 \leqq \theta_n^{(i)} < 2\pi\}, \end{aligned}$$

of width 2ε around the torus T^{n-1}.

Even if a conductivity function $\sigma^*(\sigma_1, \sigma_2, \cdots, \sigma_n)$ is not suitable, there may exist a sequence of suitable conductivity functions $\sigma_\eta^*(\sigma_1, \sigma_2, \cdots, \sigma_n)$ which converge pointwise to $\sigma^*(\sigma_1, \sigma_2, \cdots, \sigma_n)$ as $\eta \to 0$. Then $\sigma^*(\sigma_1, \sigma_2, \cdots, \sigma_n)$ can be approximated by $\sigma_\eta^*(\sigma_1, \sigma_2, \cdots, \sigma_n)$ by taking η sufficiently small. For example, the sequence of conductivity functions

$$(4.10)\qquad \sigma_\eta^*(\sigma_1, \sigma_2, \cdots, \sigma_n) = [\sigma_1\sigma_2 \cdots \sigma_n]^\eta [\sigma^*(\sigma_1, \sigma_2, \cdots, \sigma_n)]^{1-n\eta}$$

may be suitable for $1/n \geqq \eta > 0$ even when $\sigma^*(\sigma_1, \sigma_2, \cdots, \sigma_n)$ is not. In particular, for $n = 2$, the conductivity functions

$$(4.11)\qquad \sigma_\eta^* = (\sigma_1\sigma_2)^\eta [\tfrac{1}{2}(\sigma_1 + \sigma_2)]^{1-2\eta}$$

are suitable for $\frac{1}{2} \geqq \eta > 0$ even though

$$(4.12)\qquad \sigma^* = \tfrac{1}{2}(\sigma_1 + \sigma_2)$$

is not: the measures $d\mu_1(\theta_2^{(1)})$ and $d\mu_2(\theta_1^{(2)})$ of the conductivity function σ^* have Dirac delta function distributions at $\theta_2^{(1)} = \pi$ and at $\theta_1^{(2)} = \pi$ whereas the measures of σ_η^* do not. (Their distribution functions become more sharply peaked as $\eta \to 0$.)

We expect that any conductivity function can be expressed as the limit of a sequence of suitable conductivity functions. If this is true then there is no loss of generality when we restrict our analysis to the class of suitable conductivity functions.

Since there is negligible contribution to the measure in the vicinity of points where $\theta_j^{(p)} = \pi$ or 0 for some j, let us suppose that

$$(4.13)\qquad \theta_j^{(p)} \neq 0 \quad \text{or} \quad \pi \quad \text{for all} \quad j \neq p,$$

or equivalently that $h_j^{(p)} \neq 0$ or ∞ for all $j \neq p$. Thus we shall focus on finding the relation between the measures μ_q and μ_p on all regions of the torus except on those surfaces excluded by (4.13).

It follows from (3.14) and (4.6) that

$$(4.14)\qquad d\mu_j(\theta_1^{(j)}, \cdots, \theta_n^{(j)}) = (2\pi)^{-n+1} g_j(\theta_1^{(j)}, \cdots, \theta_n^{(j)})\, d\theta_1^{(j)} \cdots \theta_n^{(j)}.$$

Now the homogeneity property of the conductivity function implies

$$(4.15)\qquad m_p(h_1^{(p)}, \cdots, h_n^{(p)}) = h_q^{(p)} m_q(h_1^{(q)}, \cdots, h_n^{(q)}),$$

where the conductivity ratios $h_j^{(q)}$ are given in terms of the ratios $h_j^{(p)}$ via (4.4). Consequently from (4.6) and (4.15) we deduce that

$$(4.16)\qquad \begin{aligned} &g_p(\theta_1^{(p)}, \cdots, \theta_n^{(p)}) \\ &\quad = \lim_{\delta_1^{(p)}, \cdots, \delta_n^{(p)} \to 0^+} \mathscr{Im}\Big[\big((\tan\tfrac{1}{2}\theta_q^{(p)}) + i\delta_q^{(p)}\big) \\ &\qquad \times m_q\big((\tan\tfrac{1}{2}\theta_1^{(q)}) + i\delta_1^{(q)}, \cdots, (\tan\tfrac{1}{2}\theta_n^{(q)}) + i\delta_n^{(q)}\big)\Big] \\ &\quad = \tan\tfrac{1}{2}\theta_q^{(p)} \lim_{\delta_1^{(p)}, \cdots, \delta_n^{(p)} \to 0^+} \mathscr{Im}\Big[m_q\big((\tan\tfrac{1}{2}\theta_1^{(q)}) + i\delta_1^{(q)}, \\ &\qquad \cdots, (\tan\tfrac{1}{2}\theta_n^{(q)}) + i\delta_n^{(q)}\big)\Big], \end{aligned}$$

where the quantities

$$(4.17)\qquad \delta_j^{(q)} = \begin{cases} [\delta_j^{(p)} \tan\tfrac{1}{2}\theta_q^{(p)} - \delta_q^{(p)} \tan\tfrac{1}{2}\theta_j^{(p)}]/[\tan\tfrac{1}{2}\theta_q^{(p)}]^2 & \text{when } j \neq p, \\ -\delta_q^{(p)}/[\tan\tfrac{1}{2}\theta_q^{(p)}]^2 & \text{when } j = p, \end{cases}$$

are infinitesimal. To establish a direct correspondence between the two distribution functions $g_p(\theta_1^{(p)}, \cdots, \theta_n^{(p)})$ and $g_q(\theta_1^{(q)}, \cdots, \theta_n^{(q)})$ we need to let the positive infinitesimals $\delta_1^{(p)}, \cdots, \delta_n^{(p)}$ approach zero in such a way that the resulting infinitesimals $\delta_1^{(q)}, \cdots, \delta_n^{(q)}$ all have the same sign. This can be achieved when $\tan\frac{1}{2}\theta_q^{(p)} < 0$, i.e., when

$$(4.18)\qquad h_q^{(p)} < 0.$$

Under the conditions (4.13) and (4.18) the infinitesimals $\delta_1^{(q)}, \cdots, \delta_n^{(q)}$ will be negative provided we let $\delta_1^{(p)}, \cdots, \delta_n^{(p)}$ approach zero with

$$(4.19)\qquad \delta_j^{(p)} > \delta_q^{(p)} \tan\tfrac{1}{2}\theta_j^{(p)}/\tan\tfrac{1}{2}\theta_q^{(p)}$$

for all $j \neq q$. Then (4.16) implies

$$(4.20)\qquad g_p(\theta_1^{(p)}, \cdots, \theta_n^{(p)}) = |\tan\tfrac{1}{2}\theta_q^{(p)}| g_q(\theta_1^{(q)}, \cdots, \theta_n^{(q)}).$$

Now suppose, alternatively, that

$$h_q^{(p)} > 0. \tag{4.21}$$

Two possibilities can then occur: either $h_j^{(p)} > 0$ for all j, in which case

$$g_p\left(\theta_1^{(p)},\cdots,\theta_n^{(p)}\right) = g_q\left(\theta_1^{(q)},\cdots,\theta_n^{(q)}\right) = 0; \tag{4.22}$$

or $h_j^{(p)} < 0$ for some $j \neq q$. In the second case it follows from (4.4) that

$$h_q^{(j)} = h_q^{(p)}/h_j^{(p)} < 0 \tag{4.23}$$

and consequently (4.20) and (4.5) imply

$$\begin{aligned} g_p\left(\theta_1^{(p)},\cdots,\theta_n^{(p)}\right) &= \left|\tan\tfrac{1}{2}\theta_j^{(p)}\right| g_j\left(\theta_1^{(j)},\cdots,\theta_n^{(j)}\right) \\ &= \left|\tan\tfrac{1}{2}\theta_p^{(j)}\right|^{-1}\left|\tan\tfrac{1}{2}\theta_q^{(j)}\right| g_q\left(\theta_1^{(q)},\cdots,\theta_n^{(q)}\right) \\ &= \left|\tan\tfrac{1}{2}\theta_q^{(p)}\right| g_q\left(\theta_1^{(q)},\cdots,\theta_n^{(q)}\right). \end{aligned} \tag{4.24}$$

In summary, the relation (4.20) between the distribution functions $g_p(\theta_1^{(p)},\cdots,\theta_n^{(p)})$ and $g_q(\theta_1^{(q)},\cdots,\theta_n^{(q)})$ holds irrespective of the sign of $h_q^{(p)}$. Thus the measure μ_p is positive if and only if the measure μ_q is positive: no additional constraints are imposed on the measure by requiring that μ_p be positive for all $p \neq q$. Presumably, this result extends beyond the class of suitable conductivity functions and holds for any conductivity function.

The Fourier constraints on the measure μ_p in conjunction with (4.20) imply that, for all $p \neq q$,

$$\begin{aligned} &\int_{T^{n-1}} d\theta_1^{(p)}\, d\theta_2^{(p)} \cdots d\theta_n^{(p)} \left|\tan\tfrac{1}{2}\theta_q^{(p)}\right| \exp\left[i\left(k_1^{(p)}\theta_1^{(p)} + \cdots + k_n^{(p)}\theta_n^{(p)}\right)\right] \\ &\times g_q\left(2\tan^{-1}\left[\tan\tfrac{1}{2}\theta_1^{(p)}/\tan\tfrac{1}{2}\theta_q^{(p)}\right],\cdots,\pi-\theta_q^{(p)},\right. \\ &\qquad \left.\cdots, 2\tan^{-1}\left[\tan\tfrac{1}{2}\theta_n^{(p)}/\tan\tfrac{1}{2}\theta_q^{(p)}\right]\right) = 0 \\ &\qquad \text{unless} \quad \left(k_1^{(q)},\cdots,k_{q-1}^{(q)},k_{q+1}^{(q)},\cdots,k_n^{(q)}\right) \in \mathbf{Z}_+^{n-1} \cup \mathbf{Z}_-^{n-1}. \end{aligned} \tag{4.25}$$

This imposes a set of non-trivial constraints on the distribution function $g_q(\theta_1^{(q)},\cdots,\theta_p^{(q)},\cdots,\theta_n^{(q)})$. Because of their complexity, it is questionable whether these constraints could be used in a constructive manner. Let us therefore seek another representation of the conductivity function for which the constraints take a simpler form.

5. A Symmetric Representation Formula for the Conductivity Function

A symmetric representation, which treats all variables of the conductivity function $\sigma^*(\sigma_1, \sigma_2,\cdots,\sigma_n)$ on an equal basis, is easily obtained if we avoid using the homogeneity relation to eliminate one of the variables. We then find a representation for the function in terms of a measure on the torus T^n. The homogeneity relation gives an additional constraint on the measure.

Since the dissipation property implies $\sigma^*(\sigma_1, \sigma_2,\cdots,\sigma_n)$ has positive real part on the domain

$$H^n = \{\mathscr{Re}\,\sigma_1 > 0\} \times \{\mathscr{Re}\,\sigma_2 > 0\} \times \cdots \times \{\mathscr{Re}\,\sigma_n > 0\}, \tag{5.1}$$

let us introduce the variables

$$z_j = (1-\sigma_j)/(1+\sigma_j) \tag{5.2}$$

in terms of which

$$\sigma_j = (1-z_j)/(1+z_j). \tag{5.3}$$

These are suitable variables because the fractional linear transformation (5.2) maps the right half-plane H onto the unit disk D with the points $\sigma_j = 0$, i, ∞, and 1 getting mapped to $z_j = 1$, $-i$, -1 and 0, respectively. It follows that the function

$$f(z) = \sigma^*\left(\frac{1-z_1}{1+z_1}, \frac{1-z_2}{1+z_2},\cdots,\frac{1-z_n}{1+z_n}\right) \tag{5.4}$$

has positive real part on the unit polydisk D^n. Hence from Theorem 1 the conductivity function has the representation

$$\begin{aligned} &\sigma^*(\sigma_1, \sigma_2,\cdots,\sigma_n) \\ &= \int_{T^n}\left[-1 + 2\prod_{k=1}^{n}\frac{(1+\sigma_k)}{(1+\sigma_k)-(1-\sigma_k)\exp(-i\theta_k)}\right] d\mu(\theta_1,\theta_2,\cdots,\theta_n), \end{aligned} \tag{5.5}$$

where the measure

$$d\mu(\theta_1, \theta_2, \cdots, \theta_n)$$

$$= \lim_{\varepsilon_1, \cdots, \varepsilon_n \to 0^+} (2\pi)^{-n} \mathscr{Re}[f(\exp\{i\theta_1 - \varepsilon_1\}, \cdots, \exp\{i\theta_n - \varepsilon_n\}] \, d\theta_1 \cdots d\theta_n$$

$$(5.6) \quad = \lim_{\varepsilon_1, \cdots, \varepsilon_n \to 0^+} (2\pi)^{-n} \mathscr{Re}[\sigma^*(-i \tan\tfrac{1}{2}(\theta_1 + i\varepsilon_1), \cdots, -i \tan\tfrac{1}{2}(\theta_n + i\varepsilon_n))] \, d\theta_1 \cdots d\theta_n$$

$$= \lim_{\varepsilon_1, \cdots, \varepsilon_n \to 0^+} (2\pi)^{-n} \mathscr{Im}[\sigma^*(\tan\tfrac{1}{2}(\theta_1 + i\varepsilon_1), \cdots, \tan\tfrac{1}{2}(\theta_n + i\varepsilon_n))] \, d\theta_1 \cdots d\theta_n$$

satisfies the Fourier constraints

$$(5.7) \quad a_{k_1, k_2, \cdots, k_n} = \int_{T^n} \exp\{i(k_1\theta_1 + \cdots + k_n\theta_n)\} \, d\mu(\theta_1, \theta_2, \cdots, \theta_n) = 0 \quad \text{unless} \quad (k_1, k_2, \cdots, k_n) \in \mathbb{Z}_+^n \cup \mathbb{Z}_-^n.$$

The homogeneity relation (1.19) with $\lambda = -1$ implies

$$(5.8) \quad \sigma^*(\sigma_1, \sigma_2, \cdots, \sigma_n) = -\sigma^*(-\sigma_1, -\sigma_2, \cdots, -\sigma_n)$$

and it follows directly from (5.6) that the measure is symmetric in the sense that

$$(5.9) \quad \mu(\theta_1, \theta_2, \cdots, \theta_n) = \mu(2\pi - \theta_1, 2\pi - \theta_2, \cdots, 2\pi - \theta_n).$$

Consequently all the non-zero Fourier coefficients of the measure are real and we have

$$(5.10) \quad a_{k_1, k_2, \cdots, k_n} = \int_{T^n} \cos(k_1\theta_1 + \cdots + k_n\theta_n) \, d\mu(\theta_1, \theta_2, \cdots, \theta_n),$$

$$(5.11) \quad a_{-k_1, -k_2, \cdots, -k_n} = a_{k_1, k_2, \cdots, k_n}.$$

From the representation formula (5.5) and the symmetry (5.9) of the measure it follows that the conductivity function is necessarily real symmetric. We have

$$(5.12) \quad \begin{aligned} \overline{\sigma^*(\sigma_1, \sigma_2, \cdots, \sigma_n)} &= \int_{T^n} \left[-1 + 2\prod_{k=1}^{n} \frac{(1 + \bar{\sigma}_k)}{(1 + \bar{\sigma}_k) - (1 - \bar{\sigma}_k)\exp\{i\theta_k\}}\right] d\mu(\theta_1, \theta_2, \cdots, \theta_n) \\ &= \int_{T^n} \left[-1 + 2\prod_{k=1}^{n} \frac{(1 + \bar{\sigma}_k)}{(1 + \bar{\sigma}_k) - (1 - \bar{\sigma}_k)\exp\{-i\theta_k\}}\right] d\mu(\theta_1, \theta_2, \cdots, \theta_n) \\ &= \sigma^*(\bar{\sigma}_1, \bar{\sigma}_2, \cdots, \bar{\sigma}_n). \end{aligned}$$

The measure μ satisfies some other simple constraints. The normalization constraint (1.20) on the conductivity function together with (5.5) implies

$$(5.13) \quad \int_{T^n} d\mu(\theta_1, \theta_2, \cdots, \theta_n) = \sigma^*(1, 1, \cdots, 1) = 1.$$

Thus the measure μ has unit mass.

The wedge bound (1.24) implies σ^* is purely real when the conductivities $\sigma_1, \sigma_2, \cdots, \sigma_n$ are all purely real and non-negative or all purely real and non-positive. Consequently we have

$$(5.14) \quad d\mu(\theta_1, \theta_2, \cdots, \theta_n) = 0 \quad \text{when} \quad (\theta_1, \theta_2, \cdots, \theta_n) \in T_+^n \cup T_-^n,$$

where

$$(5.15) \quad T_+^n = \{0 \leqq \theta_1 < \pi\} \times \cdots \times \{0 \leqq \theta_n < \pi\},$$

$$(5.16) \quad T_-^n = \{\pi < \theta_1 \leqq 2\pi\} \times \cdots \times \{\pi < \theta_n \leqq 2\pi\},$$

are subregions of the torus T^n. Thus the measure vanishes on two multi-dimensional square "patches" on the torus.

To see what other constraints the homogeneity relation imposes upon the measure, let us, for simplicity, assume that the distribution function

$$(5.17) \quad g(\theta_1, \cdots, \theta_n) = \lim_{\varepsilon_1, \cdots, \varepsilon_n \to 0^+} \mathscr{Im}[\sigma^*(\tan\tfrac{1}{2}(\theta_1 + i\varepsilon_1), \cdots, \tan\tfrac{1}{2}(\theta_n + i\varepsilon_n))]$$

exists. It then clearly follows that

$$(5.18) \quad d\mu(\theta_1, \theta_2, \cdots, \theta_n) = (2\pi)^{-n} g(\theta_1, \theta_2, \cdots, \theta_n) \, d\theta_1 \, d\theta_2 \cdots d\theta_n$$

and using the homogeneity relation we deduce that, for all $\lambda \in \mathbb{R}$,

$$
\begin{aligned}
& g(\theta_1,\cdots,\theta_n) \\
(5.19)\quad & = (1/\lambda) \lim_{\varepsilon_1,\cdots,\varepsilon_n \to 0^+} \mathscr{I}m\left[\sigma^*\left(\lambda \tan\tfrac{1}{2}(\theta_1 + i\varepsilon_1),\right.\right. \\
& \qquad\qquad \left.\left.\cdots, \lambda \tan\tfrac{1}{2}(\theta_n + i\varepsilon_n)\right)\right] \\
& = (1/\lambda)\, g\left(2\tan^{-1}\left(\lambda \tan\tfrac{1}{2}\theta_1\right),\cdots, 2\tan^{-1}\left(\lambda\tan\tfrac{1}{2}\theta_n\right)\right).
\end{aligned}
$$

Thus the homogeneity relation imposes a non-trivial constraint on the distribution function. Given the distribution function at a point $(\theta_1, \theta_2, \cdots, \theta_n)$ on the torus (other than those points where $\theta_j = 0$ or π for all j) we can use (5.18) to determine the distribution function along an associated trajectory of points $(2\tan^{-1}(\lambda\tan\frac{1}{2}\theta_1),\cdots, 2\tan^{-1}(\lambda\tan\frac{1}{2}\theta_n))$ parameterized by λ.

One advantage of the symmetric representation is that the Fourier coefficients of the measure have a direct interpretation in terms of the coefficients in the perturbation expansion of the effective conductivity of a nearly homogeneous multicomponent composite. To develop this expansion, notice that the formal expression (1.16) for the effective conductivity σ^* with $\sigma_0 = -1$ reduces to

$$
\begin{aligned}
& \sigma^* = -1 + \int_Q e^* \cdot \left[\sum_{j=1}^{n} (1+\sigma_j)^{-1}\chi_j - \Gamma\right]^{-1} e^*\, dx \\
(5.20)\quad & \\
& \quad = -1 - 2\int_Q e^* \cdot \left[\Psi - \sum_{j=1}^{n} z_j\chi_j\right]^{-1} e^*\, dx,
\end{aligned}
$$

where z_j is given by (5.2), Ψ is the operator

$$(5.21)\qquad \Psi = 2\Gamma - I,$$

and we have used the fact that $I = \sum_{j=1}^{n}((1+\sigma_j)/(1+\sigma_j))\chi_j$. The operator Ψ satisfies

$$(5.22)\qquad \Psi^2 = I,$$

i.e., it is its own inverse. It follows that

$$
\begin{aligned}
& \left[\Psi - \sum_{j=1}^{n} z_j\chi_j\right]^{-1} = \Psi\left[I - \sum_{j=1}^{n} z_j\chi_j\Psi\right]^{-1} \\
(5.23)\quad & \\
& \quad = \Psi + \sum_{j=1}^{n} z_j\Psi\chi_j\Psi + \sum_{i=1}^{n}\sum_{j=1}^{n} z_iz_j\Psi\chi_i\Psi\chi_j\Psi + \cdots .
\end{aligned}
$$

By substituting this into (5.20) and noting that

$$(5.24)\qquad \Psi e^* = -e^*,$$

we obtain the perturbation expansion

$$
\begin{aligned}
& \sigma^* = 1 + 2\sum_{j=1}^{n} z_j(e^*, \chi_j\Psi e^*) + 2\sum_{i=1}^{n}\sum_{j=1}^{n} z_iz_j(e^*, \chi_i\Psi\chi_j\Psi e^*) \\
(5.25)\quad & \\
& \qquad + 2\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n} z_iz_jz_l(e^*, \chi_i\Psi\chi_j\Psi\chi_l\Psi e^*) + \cdots ,
\end{aligned}
$$

where the brackets denote the inner product

$$(5.26)\qquad (h, g) = \int_Q h(x)\cdot g(x)\, dx$$

between any two vector-valued Q-periodic functions.

From (2.10) and (5.5) we have the alternative perturbation expansion

$$(5.27)\qquad \sigma^* = \sum_{k_1, k_2,\cdots,k_n \geq 0} c_{k_1,k_2,\cdots,k_n} z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n},$$

where

$$(5.28)\qquad c_{k_1,k_2,\cdots,k_n} = \begin{cases} a_{0,0,\cdots,0} & \text{if } k_1 = \cdots = k_m = 0, \\ 2a_{k_1,k_2,\cdots,k_m} & \text{otherwise.} \end{cases}$$

By comparing this with (5.25) and equating coefficients we obtain formulae for all the Fourier coefficients. For example we find that

$$(5.29)\qquad a_{0,0,\cdots,0} = 1,$$

$$(5.30)\qquad a_{0,\cdots,0,k_j,0,\cdots,0} = \left(e^*, (\chi_j\Psi)^{k_j}e^*\right),$$

$$(5.31)\qquad a_{1,1,0,0,\cdots,0} = (e^*, \chi_1\psi\chi_2\Psi e^*) + (e^*, \chi_2\Psi\chi_1\Psi e^*),$$

and so forth.

For any medium, the first-order coefficients in (5.25) can be easily computed,

$$(5.32)\qquad (e^*, \chi_j\Psi e^*) = -p_j,$$

where p_j is the volume fraction of component j. If the medium is further assumed to be statistically isotropic, then the second order coefficients can be computed as well (see, e.g., [2], [12]),

$$(5.33)\qquad (e^*, \chi_i\Psi\chi_j\Psi e^*) = \begin{cases} p_i - \dfrac{2p_ip_j}{d}, & i \neq j, \\ p_i + \dfrac{2(p_i - p_i^2)}{d}, & i = j, \end{cases}$$

where d is the dimensionality of the system. For two-component media, (5.25) to second order becomes

$$(5.34)\qquad \begin{aligned} \sigma^* &= 1 - 2[p_1z_1 + p_2z_2] \\ &+ 2\left[z_1^2\left(p_1 + \frac{2p_1p_2}{d}\right) + z_2^2\left(p_2 + \frac{2p_1p_2}{d}\right) + z_1z_2\left(1 - \frac{4p_1p_2}{d}\right)\right] \\ &+ \cdots, \end{aligned}$$

and for three-component media,

$$(5.35)\qquad \begin{aligned} \sigma^* &= 1 - 2[p_1z_1 + p_2z_2 + p_3z_3] \\ &+ 2\left[z_1z_2\left(p_1 + p_2 - \frac{4p_1p_2}{d}\right) + z_1z_3\left(p_1 + p_3 - \frac{4p_1p_3}{d}\right)\right. \\ &+ z_2z_3\left(p_2 + p_3 - \frac{4p_2p_3}{d}\right) + z_1^2\left(p_1 + \frac{2(p_1 - p_1^2)}{d}\right) \\ &\left.+ z_2^2\left(p_2 + \frac{2(p_2 - p_2^2)}{d}\right) + z_3^2\left(p_3 + \frac{2(p_3 - p_3^2)}{d}\right)\right] + \cdots. \end{aligned}$$

Appendix

Proof of an Elementary Bound on σ^*. As mentioned in the Introduction, one of the principal uses of representation formulas for σ^* has been to derive bounds on σ^* assuming various amounts of geometrical information, as well as knowledge of the σ_i. In this appendix we prove a bound on any $f(z)$ in Theorem 1 with $\mathcal{Im}\, f(0) = 0$ which assumes knowledge of the z's, and that the mass of the measure in (2.6) is equal to 1. The extremal measures which yield this bound were used by Golden [10]–[12] to conjecture optimal bounds on σ^*. These bounds on σ^* were subsequently proven by Bergman and Milton [4], [21].

The proofs of our result here and the related bounds on σ^* are based on the "trajectory" method of Bergman [2], [4], where f as a function of m variables is treated as a function of a single variable by considering f on a (complex) one-dimensional trajectory through $\mathbb{C}^m$.

Before stating and proving our results, we would like to describe why it has some mathematical interest independent of the context of composite materials. Let $M_1^m = \{$positive measures μ on T^m satisfying the Fourier condition (2.8) and $a_{0,0,\cdots,0} = 1\}$. Since M_1^m is convex and compact (in the weak* topology over continuous functions on T^m), it has extreme points. The simplest examples of extreme points of M_1^m are of the form

$$(\text{A}.1)\qquad \mu = \delta_{\theta_1^*}(d\theta_1) \times \frac{d\theta_2}{2\pi} \times \cdots \times \frac{d\theta_m}{2\pi},$$

and permutations of these, where $\delta_{\theta_1^*}$ is a single point measure concentrated at $\theta_1^* \in [0, 2\pi)$. However, Rudin [22] and McDonald [19] have given examples of other extreme points of M_1^m, $m \geqq 2$, not of the form (A.1), and the full set of extreme points is unknown. (For $m = 1$ the only extreme points of M_1^1 are single point measures $\delta_{\theta_1^*}$, $\theta_1^* \in [0, 2\pi)$.) The bound that we are interested in is just the range B of values of $f(\tilde{z})$ in (2.6) for fixed $\tilde{z} \in D^m$ as μ varies throughout M_1^m. Now, for fixed $\tilde{z} \in D^m$, (2.6) defines a linear functional

$$(\text{A}.2)\qquad f_{\tilde{z}}(\mu) : M_1^m \to \mathbb{C}.$$

The image B of M_1^m under $f_{\tilde{z}}$ is obviously a compact convex subset of $\mathbb{C}$. However, since it is not known what all the extreme points of M_1^m are, it is not a priori clear what the extreme points of B should be. Nevertheless, we prove in a simple manner that B is a particular closed disc in the right half-plane, and furthermore, that the circular boundary C of B is parametrized (when $\tilde{z} \neq 0$ and $\max_{1 \leq j \leq m}\{|z_j|\} = |z_1|$) by the measures in (A.1) as θ_1^* varies between 0 and 2π. Thus the functional $f_{\tilde{z}}(\mu)$ sees only the simplest extreme points of M_1^m since B is the convex hull of the images of the measures in (A.1).

Without loss of generality, we take $\mathcal{Im}\, f(0) = 0$ in (2.6). We now state

THEOREM 2. *Let $\tilde{z} = (z_1, \cdots, z_m) \in D^m$, $m \geqq 1$, be fixed with $\tilde{z} \neq 0$ and $\max_{1 \leq j \leq m}\{|z_j|\} = |z_1|$. Then the range B of values of $f(z)$ in* (2.6) *(with $\mathcal{Im}\, f(0) = 0$) as μ varies in M_1^m is a closed disc in the right-half plane with boundary C which is a circle that is symmetric about the real axis and intersects it at the points a and $1/a$, where $a \equiv (1 + |z_1|)/(1 - |z_1|)$. Moreover, C is parametrized by measures of the form* (A.1) *as θ_1^* varies between* 0 *and* 2π.

Proof: For simplicity, we give the proof for D^2, but this same proof goes through essentially unchanged for D^m. For fixed $(z_1, z_2) \in D^2$, with $|z_1| \geqq |z_2|$, let $\alpha = z_2/z_1$, so that $|\alpha| \leqq 1$. Now define the following for $z \in D^1$:

$$(\text{A}.3)\qquad g(z) = f(z, \alpha z).$$

Since $\alpha z \in D^1$ when $z \in D^1$ and since f is holomorphic and has positive real part on D^2, it follows that $g(z)$ is holomorphic in D^1 and has positive real part there. Consequently (see [1]), there is a $\nu \in M^1$, the set of positive measures on $[0, 2\pi)$, such that

$$g(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \, d\nu(t). \tag{A.4}$$

By expanding both sides of (A.3) in powers of z, it is easy to see that ν belongs to the set $M_1^1 \subset M^1$, i.e., the measures in M^1 with unit mass,

$$\int_0^{2\pi} d\nu = 1. \tag{A.5}$$

For any fixed $z \in D^1$, the range $B(z)$ of $g(z)$ in (A.4) as ν varies in M_1^1 is a closed disc in the right half-plane. This follows since the set of extreme points of M_1^1 is equal to $\{\delta_{\theta^*}(dt),\ 0 \leqq \theta^* < 2\pi\}$, and the kernel in (A.4) is fractional linear in $e^{i\theta}$, with $e^{i\theta} \neq z$, for any $\theta \in [0, 2\pi)$. The boundary $C(z)$ of $B(z)$ is easily seen to be a circle which is symmetric about the real axis and intersects it at the points b and $1/b$, where $b \equiv (1 + |z|)/(1 - |z|)$. These two points are generated by the above extreme measures with $\theta^* = \arg z$ and $\theta^* = \pi + \arg z$, respectively, where "arg" denotes argument.

Now to each $\mu \in M_1^m$ there corresponds a $\nu \in M_1^1$. For any fixed $z \in D^1$, the values of $f(z, \alpha z)$, as μ in (2.6) varies in M_1^m, lie inside $B(z)$. In particular, by taking $z = z_1$, we establish that $f(z_1, z_2) \in B(z_1)$ when $|z_2| \leqq |z_1|$.

That the boundary $C(z_1)$ of $B(z_1)$ is attained by measures $\mu \in M_1^m$ is easily seen by substituting measures of the form (A.1) into (2.6). After integration we find that

$$f(z_1, z_2) = \left(1 + z_1 e^{-i\theta_1^*}\right)/\left(1 - z_1 e^{-i\theta_1^*}\right). \tag{A.6}$$

This generates the entire boundary $C(z_1)$ as θ_1^* is varied in $0 \leqq \theta_1^* < 2\pi$.

Applying Theorem 2 to σ^* in (5.5) immediately yields a bound on σ^* which assumes knowledge only of the σ_i, and nothing about the geometry. This bound, however, is not optimal. The optimal bound was conjectured by Golden [10]–[12] and was subsequently proven by Milton [21].

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