Math 2210 PRACTICE FINAL Fall 2019

1. Let $\mathbf{v} = (2, 1)$.
   (a) Rotate $\mathbf{v}$ by $90°$ counter–clockwise.
   (b) Rotate $\mathbf{v}$ by $90°$ clockwise.

2. Let $\mathbf{F} = y \sin x\mathbf{i} - z \cos x\mathbf{j} + xz^2\mathbf{k}$, and let $\varphi = 2e^{xyz}$.
   (a) Compute $\operatorname{div} \mathbf{F}$.
   (b) Compute $\operatorname{grad} \varphi$
   (c) Compute $\operatorname{curl} \mathbf{F}$ and $\nabla \times (\nabla\varphi)$.

3. Evaluate the path integral

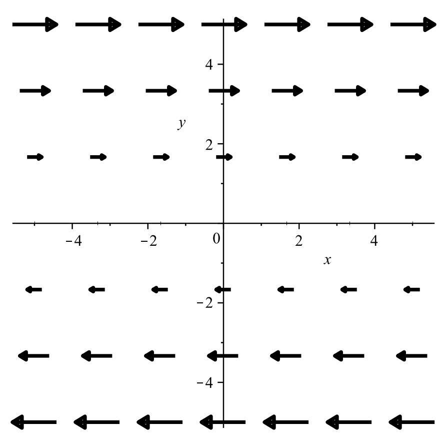
$$\oint_C (xy + 1)dx + (x^2 + y^2)dy$$

where C is the triangle with vertices $(0, 0), (1, 0)$ and $(\frac{1}{2}, 1)$.

4. Circle "T" for True or "F" for False. In the following assume $\mathbf{F} = (M, N, P)$ is a vector field where M, N and P have continuous first–order partial derivatives on an open set R, and let φ be a potential function on the same region.
   (a) T F If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path then $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is zero for every closed path C.

   (b) T F Consider φ on the unit disk D. If $\nabla \cdot (\nabla\varphi) = 0$, then the maximum of φ on D occurs at the origin $(0, 0)$.

   (c) T F The shear flow in the figure below has zero curl.



5. Use Green's Theorem to compute the area of any region S in the plane where the boundary of S, ∂S is a simple closed curve. That is show

$$A(S) = \oint_{\partial S} \mathbf{F} \cdot \mathbf{T}\, ds.$$

where $\mathbf{F} = -\frac{1}{2}y\mathbf{i} + \frac{1}{2}x\mathbf{j}$. (Hint: See Example 2 and Example 5 in §14.4 of your textbook)

6. Consider the force field $\mathbf{F}(x,y) = (2xy, x^2)$. Find

$$W = \int_{\Gamma} \mathbf{F} \cdot d\mathbf{r}\ ,$$

the work done in moving an object in this field from the origin $(0,0)$ to $(1,1)$ along the parabolic arc Γ described by the graph of $y = x^2$ connecting these two points. (Hint: Determine whether this force is conservative, and use a potential function to find the integral if it is. Otherwise, parameterize the arc and compute it directly.)

7. Let $\Omega \subset \mathbb{R}^2$ be the disk of radius 2, $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$. Consider the fluid velocity field $\mathbf{v}(x,y) = (-y^3, x^3)$ on Ω.

   (a) Find the curl of the fluid velocity, $\nabla \times \mathbf{v}$.

   (b) Use Green's Theorem to evaluate the circulation of $\mathbf{v}$ around the boundary of Ω. That is, find the line integral

   $$\oint_{\partial\Omega} \mathbf{v} \cdot d\mathbf{r}.$$

   around the circle of radius 2 forming the boundary $\partial\Omega$ of the disk Ω, traversed in the counterclockwise direction, by using your result from (a) in the area integral from Green's Theorem.

8. Suppose that an object of mass m is moving along a smooth curve C given by

$$\mathbf{r} = \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \qquad a \leq t \leq b.$$

under the influence of a conservative force $\mathbf{F}(\mathbf{r}) = \nabla f(\mathbf{r})$. From physics, we learn three facts about the object at time t

   (a) $\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t)$ (Newton's Second Law)

   (b) $KE = \frac{1}{2}m||\mathbf{r}'(t)||^2$ (kinetic energy)

   (c) $PE = -f(\mathbf{r})$ (potential energy)

Use the above to prove

$$\frac{d}{dt}(KE + PE) = 0.$$

That is that energy is conserved over time. (Hint: See page 747 of your text.)

9. Use the Divergence Theorem to evaluate the flux integral

$$\iint_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}\, dS$$

where $\mathbf{F} = (x - 2xy)\mathbf{i} + y^2\mathbf{j} + 3z\mathbf{k}$ and $\partial\Omega$ is the sphere of radius 3 centered at the origin.

10. Compute the flux through the unit sphere for the vector field $\mathbf{F}(\mathbf{r}) = \mathbf{r}$.

11. Solve Laplace's equation explicitly in one dimension,

$$u_{xx} = \frac{d^2u}{dx^2} = 0.$$

Consider a harmonic function $u(x)$ solving the above equation on the interval $[a, b]$. Demonstrate by graphing $u(x)$ that it attains its maximum and minimum values on the boundary of $[a, b]$.

12. Evaluate the surface integral

$$\iint_{G} (x^2 + y^2)\, dS$$

where G is the part of the paraboloid $z = 1 - x^2 - y^2$ that projects onto the region $R = \{(x, y) : x^2 + y^2 \leq 1\}$.

13. Let S be the solid determined by $1 \leq x^2 + y^2 + z^2 \leq 4$, and let $\mathbf{F} = x\mathbf{i} + (2y + z)\mathbf{j} + (z + x^2)\mathbf{k}$. Evaluate

$$\iint_{\partial S} \mathbf{F}\cdot\mathbf{n}\, dS$$

14. Consider the potential energy of a harmonic oscillator in three dimensions given by

$$\varphi(x, y, z) = \frac{k}{2}\left(x^2 + y^2 + z^2\right).$$

(a) Find the force $\mathbf{F}(x, y, z) = -\nabla\varphi$ at a given position $\mathbf{r} = (x, y, z)$.

(b) What are the equipotential surfaces (level sets)?

(c) Find the divergence and curl of $\mathbf{F}$, that is, find $\nabla\cdot\mathbf{F}$ and $\nabla\times\mathbf{F}$.

(d) Use the divergence theorem for $\mathbf{F}$ over $\Omega = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq a^2\}$, the ball of radius a, to find the flux

$$\iint_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}\, dS$$

of $\mathbf{F}$ through the spherical surface $\partial\Omega = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = a^2\}$.

15. Let $\mathbf{F}(x, y, z) = (2x, 2y, 2z)$. Find a scalar function $\varphi(x, y, z)$ such that $\mathbf{F} = \nabla\varphi$.