Mathematics 2210 **PRACTICE EXAM II** **Fall 2019**

1. Suppose you have a function $z = f(x, y)$ whose graph is a surface in $\mathbb{R}^3$. Describe how the level sets of the function relate geometrically to the surface. What is the relationship between the level sets and the gradient of f, ∇f?

2. Consider the paraboloid defined by $z = f(x, y) = (x - 2)^2 + (y - 2)^2$.

   (a) Sketch the paraboloid.

   (b) On a separate set of xy axes, sketch the level curves $z = 1$ and $z = \sqrt{2}$.

   (c) On the same axes as above, draw the gradient vector at the point $(2, 0)$.

   (d) Find the global extrema of f on $\mathbb{R}^2$ and verify your results using the second partial derivative test.

3. Suppose that the temperature in $\mathbb{R}^3$ is given by

$$T(x, y, z) = \frac{1}{1 + x^2 + y^2 + z^2},$$

   and further suppose that your position is given by the curve:

$$\mathbf{r}(t) = (x(t), y(t), z(t)) = (2t, 4t^2, 1).$$

   (a) Use the chain rule to find the rate of change $\frac{dT}{dt}$ of the temperature T with respect to time t, as you travel along the curve given above. Express your answer in terms of t only and simplify it.

   (b) Find the direction in which the temperature is increasing the fastest at time $t = 2$.

4. Consider the function $f(x, y) = x^2 - xy^3$.

   (a) If $x = \cos(t)$ and $y = \sin(t)$, find $\frac{df}{dt}$.

   (b) Find the differential df at the point $(1, 1)$ if x increases by 0.1 and y decreases by 0.2.

5. Find the following limit. If it does not exist, demonstrate why not.

$$\lim_{(x,y)\to(0,0)} \frac{x - 7y}{x + y}$$

6. Find the following limit. If it does not exist, demonstrate why not.

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{y}$$

7. Find the following limit. If it does not exist, demonstrate why not.

$$\lim_{(x,y)\to(0,0)} \frac{1 - \cos\sqrt{x^2 + y^2}}{x^2 + y^2}$$

8. Find the directional derivative of $f(x, y, z) = (x^2 - y^2)e^{2z}$,
   (a) at the point $P = (1, 2, 0)$ in the direction $2\mathbf{i} + \mathbf{j} + \mathbf{k}$.
   (b) At the point P, find the direction of maximal increase of f.

9. Consider the surface defined by $f(x, y, z) = xe^y + ye^z + ze^x = 0$.
   (a) Find the gradient of f.
   (b) Find the equation for the tangent plane at the point $(0, 0, 0)$.
   (c) Find the directional derivative of f in the direction $\mathbf{i} + \mathbf{j}$ at the point $(0, 0, 0)$.

10. Find the local maxima, minima, and saddle points of the function $f(x, y) = x^2 + y^2 - 3xy$.

11. Show that $u(x, t) = \cos(x - ct) + \sin(x - ct)$ solves the wave equation:

$$c^2 \, u_{xx} = u_{tt} \qquad \text{OR} \qquad c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}.$$

12. Consider the saddle function $f(x, y) = x^2 - y^2$.
    (a) Show that this function is harmonic.
    (b) Now consider this function on the unit disk $D = \{(x, y) : x^2 + y^2 \leq 1\}$. Find the global extrema of f on the disk D.

13. Let $\phi(x, y)$ be the electric potential due to a point charge in two dimensions, that is, $\phi(x, y) = k \ln r$, where $r = \sqrt{x^2 + y^2}$ and you may take $k = -1$. (a) Find the level curves of ϕ and its gradient $\vec{E} = -\nabla\phi$. Sketch $\vec{E}$ at the points $(1, 0), (0, 1), (-1, 0), (0, -1)$ and interpret its meaning. (b) Find the level sets for $\phi(x, y, z) = mgz$ in three dimensions, find $\vec{F} = -\nabla\phi$, and interpret the meaning of $\vec{F}$.