CHAPTER 10

Fourier Series, Integrals, and Transforms

Fourier series[1] (Sec. 10.2) are series of cosine and sine terms and arise in the important practical task of representing general periodic functions. They constitute a very important tool in solving problems that involve ordinary and partial differential equations. In the present chapter we discuss these series and their engineering use from a practical viewpoint. Further applications follow in the next chapter on partial differential equations.

The *theory* of Fourier series is rather complicated, but the *application* of these series is simple. Fourier series are, in a certain sense, more universal than Taylor series, because many *discontinuous* periodic functions of practical interest can be developed in Fourier series, but, of course, do not have Taylor series representations.

The last four sections (10.8–10.11) concern **Fourier integrals** and **Fourier transforms,** which extend the ideas and techniques of Fourier series to nonperiodic functions defined for all x. (Corresponding applications to partial differential equations will be considered in the next chapter, in Sec. 11.6.

Prerequisite for this chapter: Elementary integral calculus.
Sections that may be omitted in a shorter course: 10.5–10.10.
References: Appendix 1, Part C.
Answers to problems: Appendix 2.

[1]JEAN-BAPTISTE JOSEPH FOURIER (1768—1830), French physicist and mathematician, lived and taught in Paris, accompanied Napoleon to Egypt, and was later made prefect of Grenoble. He utilized Fourier series in his main work *Théorie analytique de la chaleur* (*Analytic Theory of Heat*, Paris 1822), in which he developed the theory of heat conduction (heat equation, see Sec. 11.5). These new series became a most important tool in mathematical physics and also had considerable influence on the further development of mathematics itself; see Ref. [9] in Appendix 1.

10.1 Periodic Functions. Trigonometric Series

A function $f(x)$ is called **periodic** if it is defined for all[2] real x and if there is some positive number p such that

$$f(x + p) = f(x) \qquad \text{for all } x. \tag{1}$$

This number p is called a **period** of $f(x)$. The graph of such a function is obtained by periodic repetition of its graph in any interval of length p (Fig. 236). Periodic phenomena and functions have many applications, as was mentioned before.

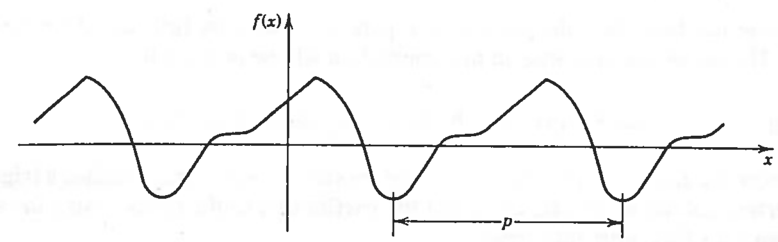


Fig. 236. Periodic function

Familiar periodic functions are the sine and cosine functions. We note that the function $f = c = const$ is also a periodic function in the sense of the definition, because it satisfies (1) for every positive p. Examples of functions that are *not* periodic are x, x^2, x^3, e^x, $\cosh x$, and $\ln x$, to mention just a few.

From (1) we have $f(x + 2p) = f[(x + p) + p] = f(x + p) = f(x)$, etc., and for any integer n,

$$f(x + np) = f(x) \qquad \text{for all } x. \tag{2}$$

Hence $2p, 3p, 4p, \cdots$ are also periods of $f(x)$. Furthermore, if $f(x)$ and $g(x)$ have period p, then the function

$$h(x) = af(x) + bg(x) \qquad (a, b \text{ constant})$$

also has the period p.

If a periodic function $f(x)$ has a smallest period p (> 0), this is often called the **fundamental period** of $f(x)$. For $\cos x$ and $\sin x$ the fundamental period is 2π, for $\cos 2x$ and $\sin 2x$ it is π, and so on. A function without fundamental period is $f = const$.

Trigonometric Series

Our problem in the first few sections of this chapter will be the representation of various functions of period $p = 2\pi$ in terms of the simple functions

$$1, \quad \cos x, \quad \sin x, \quad \cos 2x, \quad \sin 2x, \cdots, \quad \cos nx, \quad \sin nx, \cdots. \tag{3}$$

[2]Except perhaps for certain isolated x, such as $\pm\pi/2, \pm 3\pi/2, \cdots$ in the case of $\tan x$ (which is periodic with period π).

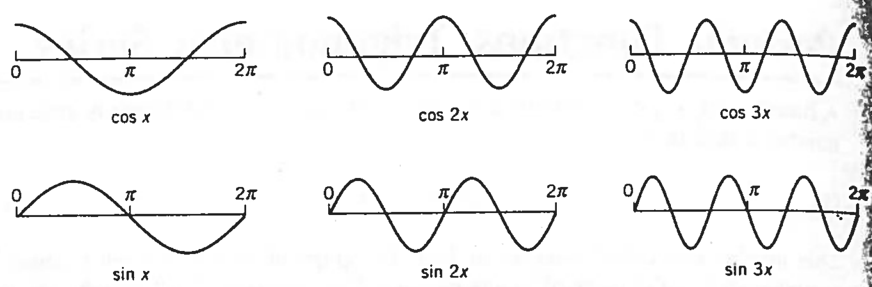


Fig. 237. Cosine and sine functions having the period 2π

These functions have the period 2π. Figure 237 shows the first few of them.

The series that will arise in this connection will be of the form

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots, \tag{4}$$

where $a_0, a_1, a_2, \cdots, b_1, b_2, \cdots$ are real constants. Such a series is called a **trigonometric series,** and the a_n and b_n are called the **coefficients** of the series. Using the summation sign,[3] we may write this series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \tag{4}$$

The set of functions (3) from which we have made up the series (4) is often called the **trigonometric system,** to have a short name for it.

We see that each term of the series (4) has the period 2π. Hence *if the series (4) converges, its sum will be a function of period* 2π.

The point is that trigonometric series can be used for representing any practically important periodic function f, simple or complicated, of any period p. (This series will then be called the *Fourier series* of f.)

PROBLEM SET 10.1

Fundamental Period. Find the smallest positive period p of

1. $\cos x,\quad \sin x,\quad \cos 2x,\quad \sin 2x,\quad \cos \pi x,\quad \sin \pi x,\quad \cos 2\pi x,\quad \sin 2\pi x$
2. $\cos nx,\quad \sin nx,\quad \cos \frac{2\pi x}{k},\quad \sin \frac{2\pi x}{k},\quad \cos \frac{2\pi n x}{k},\quad \sin \frac{2\pi n x}{k}$
3. **(Vector space)** If $f(x)$ and $g(x)$ have period p, show that $h = af + bg$ (a, b constant) has the period p. Thus all functions of period p form a vector space.
4. **(Integer multiples of period)** If p is a period of $f(x)$, show that np, $n = 2, 3, \cdots$, is a period of $f(x)$.

[3] And inserting parentheses; from a convergent series this gives again a convergent series with the same sum, as can be proved.

5. **(Constant)** Show that the function $f(x) = const$ is a periodic function of period p for every positive p.
6. **(Change of scale)** If $f(x)$ is a periodic function of x of period p, show that $f(ax)$, $a \neq 0$, is a periodic function of x of period p/a, and $f(x/b)$, $b \neq 0$, is a periodic function of x of period bp. Verify these results for $f(x) = \cos x$, $a = b = 2$.

Graphs of 2π-Periodic Functions

Sketch or plot the following functions $f(x)$, which are assumed to be periodic with period 2π and, for $-\pi < x < \pi$, are given by the formulas

7. $f(x) = x$
8. $f(x) = x^2$
9. $f(x) = |x|$
10. $f(x) = \pi - |x|$
11. $f(x) = |\sin x|$
12. $f(x) = e^{-|x|}$
13. $f(x) = \begin{cases} x & \text{if } -\pi \leqq x \leqq 0 \\ 0 & \text{if } 0 \leqq x \leqq \pi \end{cases}$
14. $f(x) = \begin{cases} 0 & \text{if } -\pi \leqq x \leqq 0 \\ x^2 & \text{if } 0 \leqq x \leqq \pi \end{cases}$
15. $f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0 \\ 1 & \text{if } 0 < x < \pi \end{cases}$
16. $f(x) = \begin{cases} x & \text{if } -\pi < x < 0 \\ \pi - x & \text{if } 0 < x < \pi \end{cases}$
17. $f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ e^{-x} & \text{if } 0 < x < \pi \end{cases}$
18. $f(x) = \begin{cases} x^2 & \text{if } -\pi < x < 0 \\ -x^2 & \text{if } 0 < x < \pi \end{cases}$
19. **CAS PROJECT. Plotting Periodic Functions.** (a) Write a program for plotting periodic functions $f(x)$ of period 2π given for $-\pi < x \leqq \pi$. Using your program, plot the functions in Probs. 7–12 for $-10\pi \leqq x \leqq 10\pi$. Also plot some functions of your own choice.
    (b) Extend your program to 2π-periodic functions given on two subintervals of the same length, as in Probs. 13–18. Apply your program to those problems with $-10\pi \leqq x \leqq 10\pi$.
20. **CAS PROJECT. Partial Sums of Trigonometric Series.** (a) Write a program that prints a partial sum[4] of a trigonometric series (4). Applying it, list all partial sums of up to five nonzero terms of each of the series

$$\frac{1}{3}\pi^2 - 4\left(\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - \frac{1}{16}\cos 4x + - \cdots\right)$$

$$\frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \cdots\right)$$

$$2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + - \cdots\right).$$

   (b) Plot the partial sums in (a) (for each series on common axes). Guess what periodic function the series might represent.

10.2 Fourier Series

Fourier series arise from the practical task of representing a given periodic function $f(x)$ in terms of cosine and sine functions. These series are trigonometric series (Sec. 10.1) whose coefficients are determined from $f(x)$ by the "Euler formulas" [(6), below], which we shall derive first. Afterwards we shall take a look at the theory of Fourier series.

[4] That is, $a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$ for $N = 1, 2, 3, \cdots$.

Euler Formulas for the Fourier Coefficients

Let us assume that $f(x)$ is a periodic function of period 2π and is integrable over a period. Let us further assume that $f(x)$ can be **represented** by a trigonometric series,

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx); \tag{1}$$

that is, we assume that this series converges and has $f(x)$ as its sum. Given such a function $f(x)$, we want to determine the coefficients a_n and b_n of the corresponding series (1).

Determination of the constant term a_0. Integrating on both sides of (1) from $-\pi$ to π, we get

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\right] dx.$$

If term-by-term integration of the series is allowed,[5] we obtain

$$\int_{-\pi}^{\pi} f(x)\,dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx\,dx + b_n \int_{-\pi}^{\pi} \sin nx\,dx\right).$$

The first term on the right equals $2\pi a_0$. All the other integrals on the right are zero, as can be readily seen by integration. Hence our first result is

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx. \tag{2}$$

Determination of the coefficients a_n of the cosine terms. Similarly, we multiply (1) by $\cos mx$, where m is any ***fixed*** positive integer, and integrate from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x) \cos mx\,dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\right] \cos mx\,dx. \tag{3}$$

Integrating term by term, we see that the right side becomes

$$a_0 \int_{-\pi}^{\pi} \cos mx\,dx + \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \cos nx \cos mx\,dx + b_n \int_{-\pi}^{\pi} \sin nx \cos mx\,dx\right].$$

The first integral is zero. By applying (11) in Appendix A3.1 we obtain

$$\int_{-\pi}^{\pi} \cos nx \cos mx\,dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos (n+m)x\,dx + \frac{1}{2} \int_{-\pi}^{\pi} \cos (n-m)x\,dx,$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx\,dx = \frac{1}{2} \int_{-\pi}^{\pi} \sin (n+m)x\,dx + \frac{1}{2} \int_{-\pi}^{\pi} \sin (n-m)x\,dx.$$

[5] This is justified, for instance, in the case of uniform convergence (see Theorem 3 in Sec. 14.5).

Integration shows that the four terms on the right are zero, except for the last term in the first line, which equals π when $n = m$. Since in (3) this term is multiplied by a_m, the right side in (3) equals $a_m\pi$. Our second result is

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx\,dx, \qquad m = 1, 2, \cdots. \tag{4}$$

Determination of the coefficients b_n of the sine terms. We finally multiply (1) by $\sin mx$, where m is any ***fixed*** positive integer, and then integrate from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x) \sin mx\,dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\right] \sin mx\,dx. \tag{5}$$

Integrating term by term, we see that the right side becomes

$$a_0 \int_{-\pi}^{\pi} \sin mx\,dx + \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \cos nx \sin mx\,dx + b_n \int_{-\pi}^{\pi} \sin nx \sin mx\,dx\right].$$

The first integral is zero. The next integral is of the kind considered before, and is zero for all $n = 1, 2, \cdots$. For the last integral we obtain

$$\int_{-\pi}^{\pi} \sin nx \sin mx\,dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos (n-m)x\,dx - \frac{1}{2} \int_{-\pi}^{\pi} \cos (n+m)x\,dx.$$

The last term is zero. The first term on the right is zero when $n \neq m$ and is π when $n = m$. Since in (5) this term is multiplied by b_m, the right side in (5) is equal to $b_m\pi$, and our last result is

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx\,dx, \qquad m = 1, 2, \cdots.$$

Summary of These Calculations: Fourier Coefficients, Fourier Series

From (2), (4), and the formula just obtained, writing n in place of m, we have the so-called **Euler[6] formulas**

(6)

(a) $a_0 = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f(x)\,dx$

(b) $a_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x) \cos nx\,dx \qquad n = 1, 2, \cdots,$

(c) $b_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x) \sin nx\,dx \qquad n = 1, 2, \cdots.$

[6] See footnote 9 in Sec. 2.6.

These numbers given by (6) are called the **Fourier coefficients** of $f(x)$. The trigonometric series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{7}$$

with coefficients given in (6) is called the **Fourier series** of $f(x)$ (regardless of convergence—we shall discuss this later in this section).

EXAMPLE 1 Rectangular wave

Find the Fourier coefficients of the periodic function $f(x)$ in Fig. 238*a*. The formula is

$$f(x) = \begin{cases} -k & \text{if} \quad -\pi < x < 0 \\ k & \text{if} \quad 0 < x < \pi \end{cases} \qquad \text{and} \qquad f(x + 2\pi) = f(x).$$

Functions of this kind occur as external forces acting on mechanical systems, electromotive forces in electric circuits, etc. (The value of $f(x)$ at a single point does not affect the integral; hence we can leave $f(x)$ undefined at $x = 0$ and $x = \pm\pi$.)

Solution. From (6a) we obtain $a_0 = 0$. This can also be seen without integration, since the area under the curve of $f(x)$ between $-\pi$ and π is zero. From (6b),

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx\, dx + \int_{0}^{\pi} k\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[-k\frac{\sin nx}{n}\Bigg|_{-\pi}^{0} + k\frac{\sin nx}{n}\Bigg|_{0}^{\pi}\right] = 0$$

because $\sin nx = 0$ at $-\pi$, 0, and π for all $n = 1, 2, \cdots$. Similarly, from (6c) we obtain

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx\, dx + \int_{0}^{\pi} k\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[k\frac{\cos nx}{n}\Bigg|_{-\pi}^{0} - k\frac{\cos nx}{n}\Bigg|_{0}^{\pi}\right].$$

Since $\cos(-\alpha) = \cos\alpha$ and $\cos 0 = 1$, this yields

$$b_n = \frac{k}{n\pi}[\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0] = \frac{2k}{n\pi}(1 - \cos n\pi).$$

Now, $\cos\pi = -1$, $\cos 2\pi = 1$, $\cos 3\pi = -1$, etc.; in general,

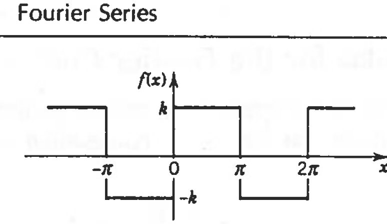
$$\cos n\pi = \begin{cases} -1 & \text{for odd } n, \\ 1 & \text{for even } n, \end{cases} \qquad \text{and thus} \qquad 1 - \cos n\pi = \begin{cases} 2 & \text{for odd } n, \\ 0 & \text{for even } n. \end{cases}$$

Hence the Fourier coefficients b_n of our function are

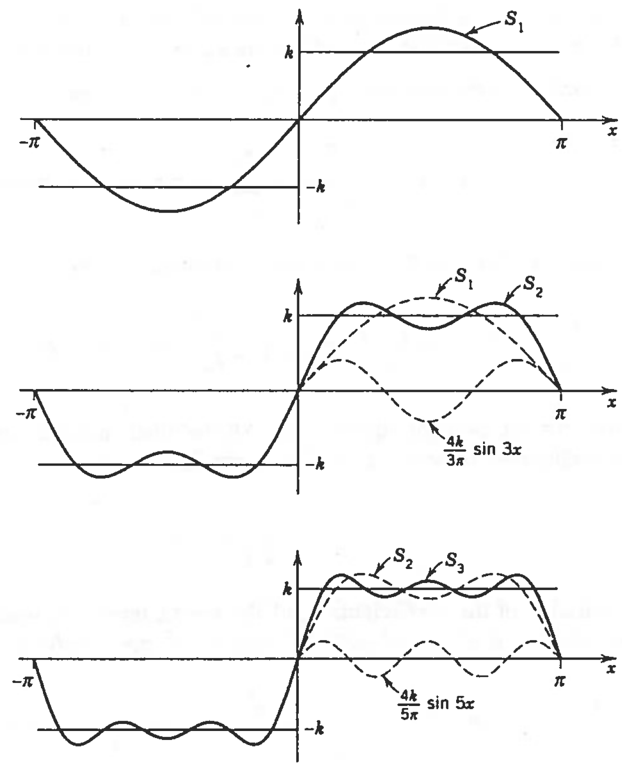
$$b_1 = \frac{4k}{\pi}, \qquad b_2 = 0, \qquad b_3 = \frac{4k}{3\pi}, \qquad b_4 = 0, \qquad b_5 = \frac{4k}{5\pi}, \cdots.$$

Since the a_n are zero, the Fourier series of $f(x)$ is

$$\frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right). \tag{8}$$



(*a*) The given function $f(x)$ (Periodic square wave)



(*b*) The first three partial sums of the corresponding Fourier series

Fig. 238. Example 1

The partial sums are

$$S_1 = \frac{4k}{\pi}\sin x, \qquad S_2 = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right), \qquad \text{etc.,}$$

Their graphs in Fig. 238 seem to indicate that the series is convergent and has the sum $f(x)$, the given function. We notice that at $x = 0$ and $x = \pi$, the points of discontinuity of $f(x)$, all partial sums have the value zero, the arithmetic mean of the values $-k$ and k of our function.

Furthermore, assuming that $f(x)$ is the sum of the series and setting $x = \pi/2$, we have

$$f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - + \cdots\right),$$

thus

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots = \frac{\pi}{4}.$$

This is a famous result by Leibniz (obtained in 1673 from geometrical considerations). It illustrates that the values of various series with constant terms can be obtained by evaluating Fourier series at specific points. ◀

Orthogonality of the Trigonometric System

The **trigonometric system** (3), Sec. 10.1,

$$1, \quad \cos x, \quad \sin x, \quad \cos 2x, \quad \sin 2x, \quad \cdots, \quad \cos nx, \quad \sin nx, \quad \cdots$$

is **orthogonal** *on the interval* $-\pi \leqq x \leqq \pi$ (hence on any interval of length 2π, because of periodicity). By definition, this means that the integral of the product of any two different of these functions over that interval is zero; in formulas, for any integers m and $n \neq m$ we have

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0 \qquad (m \neq n)$$

and

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0 \qquad (m \neq n),$$

and for any integers m and n (including $m = n$) we have

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0.$$

This is the most important property of the trigonometric system, the key in deriving the Euler formulas (where we proved this orthogonality).

Convergence and Sum of Fourier Series

Throughout this chapter we consider Fourier series from a practical point of view. We shall see that the application of these series is rather simple. In contrast to this, their theory is complicated, and we shall not go into any details of it. However, we present a theorem on the convergence and the sum of Fourier series that takes care of most applications.

Suppose that $f(x)$ is any given periodic function of period 2π for which the integrals in (6) exist; for instance, $f(x)$ is continuous or merely piecewise continuous (continuous except for finitely many finite jumps in the interval of integration). Then we can compute the Fourier coefficients (6) of $f(x)$ and use them to form the Fourier series (7) of $f(x)$. It would be nice if the series thus obtained converged and had the sum $f(x)$. Most functions appearing in applications are such that this is true (except at jumps of $f(x)$, which we discuss below). In this case, in which the Fourier series of $f(x)$ does represent $f(x)$, we write

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

with an equality sign. If the Fourier series of $f(x)$ does not have the sum $f(x)$ or does not converge, one still writes

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

with a tilde $\sim$, which indicates that the trigonometric series on the right has the Fourier coefficients of $f(x)$ as its coefficients, so it is the Fourier series of $f(x)$.

The class of functions that can be represented by Fourier series is surprisingly large and general. Corresponding sufficient conditions covering almost any conceivable application are as follows.

THEOREM 1 (Representation by a Fourier series)

If a periodic function $f(x)$ with period 2π is piecewise continuous[7] in the interval $-\pi \leqq x \leqq \pi$ and has a left-hand derivative and right-hand derivative[8] at each point of that interval, then the Fourier series (7) of $f(x)$ [with coefficients (6)] is convergent. Its sum is $f(x)$, except at a point x_0 at which $f(x)$ is discontinuous and the sum of the series is the average of the left- and right-hand limits[8] of $f(x)$ at x_0.

PROOF OF CONVERGENCE IN THEOREM 1. We prove convergence for a continuous function $f(x)$ having continuous first and second derivatives. Integrating (6b) by parts, we get

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx = \left.\frac{f(x)\sin nx}{n\pi}\right|_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin nx \, dx.$$

The first term on the right is zero. Another integration by parts gives

$$a_n = \left.\frac{f'(x)\cos nx}{n^2\pi}\right|_{-\pi}^{\pi} - \frac{1}{n^2\pi}\int_{-\pi}^{\pi} f''(x)\cos nx \, dx.$$

The first term on the right is zero because of the periodicity and continuity of $f'(x)$. Since f'' is continuous in the interval of integration, we have

$$|f''(x)| < M$$

for an appropriate constant M. Furthermore, $|\cos nx| \leqq 1$. It follows that

$$|a_n| = \frac{1}{n^2\pi}\left|\int_{-\pi}^{\pi} f''(x)\cos nx \, dx\right| < \frac{1}{n^2\pi}\int_{-\pi}^{\pi} M \, dx = \frac{2M}{n^2}.$$

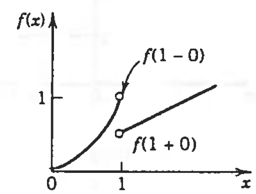


Fig. 239. Left- and right-hand limits

$f(1 - 0) = 1,$

$f(1 + 0) = \frac{1}{2}$

of the function

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ x/2 & \end{cases}$$

[7]Definition in Sec. 5.1.

[8]The **left-hand limit** of $f(x)$ at x_0 is defined as the limit of $f(x)$ as x approaches x_0 from the left and is frequently denoted by $f(x_0 - 0)$. Thus

$$f(x_0 - 0) = \lim_{h \to 0} f(x_0 - h) \text{ as } h \to 0 \text{ through positive values.}$$

The **right-hand limit** is denoted by $f(x_0 + 0)$ and

$$f(x_0 + 0) = \lim_{h \to 0} f(x_0 + h) \text{ as } h \to 0 \text{ through positive values.}$$

The **left-** and **right-hand derivatives** of $f(x)$ at x_0 are defined as the limits of

$$\frac{f(x_0 - h) - f(x_0 - 0)}{-h} \quad \text{and} \quad \frac{f(x_0 + h) - f(x_0 + 0)}{h},$$

respectively, as $h \to 0$ through positive values. Of course if $f(x)$ is continuous at x_0, the last term in both numerators is simply $f(x_0)$.

Similarly, $|b_n| < 2\,M/n^2$ for all n. Hence the absolute value of each term of the Fourier series of $f(x)$ is at most equal to the corresponding term of the series

$$|a_0| + 2M\left(1 + 1 + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \cdots\right)$$

which is convergent. Hence that Fourier series converges and the proof is complete. (Readers already familiar with uniform convergence will see that, by the Weierstrass test in Sec. 14.5, under our present assumptions the Fourier series converges uniformly, and our derivation of (6) by integrating term by term is then justified by Theorem 3 of Sec. 14.5.)

The proof of convergence in the case of a piecewise continuous function $f(x)$ and the proof that under the assumptions in the theorem the Fourier series (7) with coefficients (6) represents $f(x)$ are substantially more complicated; see, for instance, Ref. [C9]. ◀

EXAMPLE 2 Convergence at a jump as indicated in Theorem 1

The square wave in Example 1 has a jump at $x = 0$. Its left-hand limit there is $-k$ and its right-hand limit is k (Fig. 238). Hence the average of these limits is 0. The Fourier series (8) of the square wave does indeed converge to this value when $x = 0$ because then all its terms are 0. Similarly for the other jumps. This is in agreement with Theorem 1. ◀

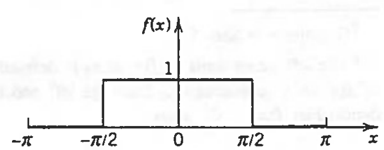
Summary. A Fourier series of a given function $f(x)$ of period 2π is a series of the form (7) with coefficients given by the Euler formulas (6). Theorem 1 gives conditions that are sufficient for this series to converge and at each x to have the value $f(x)$, except at discontinuities of $f(x)$, where the series equals the arithmetic mean of the left-hand and right-hand limits of $f(x)$ at that point.

PROBLEM SET 10.2

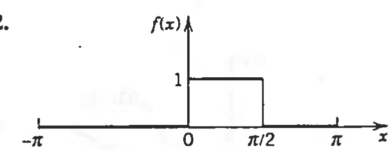
Fourier Series

Showing the details of your work, find the Fourier series of the function $f(x)$, which is assumed to have the period 2π, and plot accurate graphs of the first three partial sums, where $f(x)$ equals

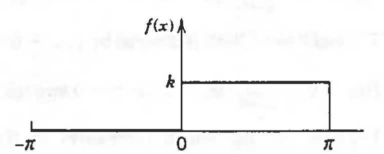
1.



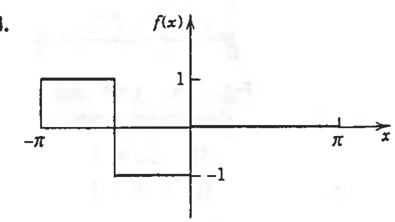
2.



3.



4.



5. $f(x) = x \quad (-\pi < x < \pi)$
6. $f(x) = x \quad (0 < x < 2\pi)$
7. $f(x) = x^2 \quad (-\pi < x < \pi)$
8. $f(x) = x^2 \quad (0 < x < 2\pi)$
9. $f(x) = x^3 \quad (-\pi < x < \pi)$
10. $f(x) = x + |x| \quad (-\pi < x < \pi)$

11. $f(x) = \begin{cases} 1 & \text{if } -\pi < x < 0 \\ -1 & \text{if } 0 < x < \pi \end{cases}$
12. $f(x) = \begin{cases} -1 & \text{if } 0 < x < \pi/2 \\ 0 & \text{if } \pi/2 < x < 2\pi \end{cases}$
13. $f(x) = \begin{cases} 1 & \text{if } -\pi/2 < x < \pi/2 \\ -1 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$
14. $f(x) = \begin{cases} x & \text{if } -\pi/2 < x < \pi/2 \\ \pi - x & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$
15. $f(x) = \begin{cases} x & \text{if } -\pi/2 < x < \pi/2 \\ 0 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$
16. $f(x) = \begin{cases} x^2 & \text{if } -\pi/2 < x < \pi/2 \\ \pi^2/4 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$
17. **(Discontinuity)** Verify the last statement in Theorem 1 regarding discontinuities for the function in Prob. 1.
18. **CAS (Orthogonality).** Integrate and plot a typical integral, for instance, that of $\sin 3x \sin 4x$, from $-a$ to a, as a function of a, and conclude orthogonality of $\sin 3x$ and $\sin 4x$ for $a = \pi$ from the plot.
19. **CAS PROJECT. Fourier Series. (a)** Write a program for obtaining any partial sum of a Fourier series (7).

    (b) Using the program, list all partial sums of up to five nonzero terms of the Fourier series in Probs. 5, 11, and 15, and make three corresponding plots. Comment on the accuracy.
20. **(Calculus review)** Review integration techniques for integrals as they may arise from the Euler formulas, for instance, definite integrals of $x \sin nx$, $x^2 \cos nx$, $e^{-x} \sin nx$, etc.

Answers to Odd-Numbered Problems

PROBLEM SET 10.1, page 528

1. $2\pi, 2\pi, \pi, \pi, 2, 2, 1, 1$

PROBLEM SET 10.2, page 536

1. $\frac{1}{2} + \frac{2}{\pi}\left(\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - + \cdots\right)$

3. $\frac{k}{2} + \frac{2k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$

5. $2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + - \cdots\right)$

7. $\frac{\pi^2}{3} - 4\left(\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - \frac{1}{16}\cos 4x + - \cdots\right)$

9. $2\left[\left(\frac{\pi^2}{1} - \frac{6}{1^3}\right)\sin x - \left(\frac{\pi^2}{2} - \frac{6}{2^3}\right)\sin 2x + \left(\frac{\pi^2}{3} - \frac{6}{3^3}\right)\sin 3x - + \cdots\right]$

11. $-\frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$

13. $\frac{4}{\pi}\left(\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - + \cdots\right)$

15. $\frac{2}{\pi}\sin x + \frac{1}{2}\sin 2x - \frac{2}{9\pi}\sin 3x - \frac{1}{4}\sin 4x + \frac{2}{25\pi}\sin 5x + \cdots$