Problem 1

Explain whether the following statements are true or false, and either prove them (or carefully quote a theorem), or give a counterexample. You cannot ask about this problem during office hours :)  

a) There exists a Markov Chain such that all states are transient.  
b) If we cannot find an invariant distribution for a Markov chain, it means that the chain is transient.  
c) If we can find an invariant distribution for an irreducible, aperiodic Markov Chain, then the chain is recurrent.  
d) If for an irreducible, aperiodic Markov Chain on $\mathbb{Z}^d$ we can show that $\mathbb{E}T_{0,0}^1 < +\infty$, we can conclude that the chain is not null recurrent. 

Problem 2

Let $X_n$ be a Markov Chain on the state space $\{0, 1, 2, \ldots\}$ with the transition probabilities  

\[
\begin{align*}
p(x, x + 1) &= p_x \quad \text{for } x \geq 0 \\
p(x, x - 1) &= q_x \quad \text{for } x > 0 \\
p(x, x) &= r_x \quad \text{for } x \geq 0
\end{align*}
\]

while the other $p(x, y) = 0$. Note then that $p_x + q_x + r_x = 1$. Let $T_y = \min\{n \geq 0 : X_n = y\}$ be the time of the first visit to $y$ and let $h_M(x) = \mathbb{P}\{T_M < T_0 | X_0 = x\}$, i.e. the probability that, if we start at $x$, we reach $M$ before reaching $0$. 

(1) What are $h_M(M + 3)$, $h_M(0)$, $h_M(M)$?  
(2) Do a first step analysis to SHOW that  

\[h_M(x) = p_x h_M(x + 1) + r_x h_N(x) + q_x h_M(x - 1).\]

Then, set $q_x = q, p_x = p, r_x = r$ to solve the recursion.  
(3) Is it necessary to have an infinite chain if all we care is $h_M(x)$? Why?  
(4) Check whether this chain is irreducible, and after that show (using the solution in (2) ) that is recurrent if and only if $\sum_{y=1}^{\infty} \prod_{x=1}^{y-1} q/p = \infty$ where by convention $\prod_{x=1}^{0} = 1$. 

Problem 3

Let a sequence of non-negative numbers $\{p_i\}_{1 \leq i < \infty}$ such that $\sum_i p_i = 1$. Consider a Markov Chain on the non-negative integers that does the following: If you are at state 0 you jump to state $i$ with probability $p_i$. If you are at any other state $x > 0$ you jump with probability 1 to state $x - 1$ and keep moving to the left until you reach 0 where this process restarts.  

(1) Explain why this chain is recurrent.
(2) Compute \( E(T_{0,0}^1) \).
(3) Find a condition on the \( p_i \)'s so that the chain is positive recurrent.
(4) When the chain is positive recurrent, it has an invariant probability \( \pi \). Find \( \pi(0) \).
(5) In the case where the chain is positive recurrent, find \( \pi \) in terms of \( \pi(0) \).

Problem 4
Consider a population of animals in a very hostile environment, where each individual may die before is able to reproduce with probability \( p \). If the individual survives, it gives birth to either one or two offsprings with equal probability, and cannot reproduce after that, because of lack of food.

(1) For which values of \( p \), the population is eventually going to go extinct?
(2) If \( p = 0.9 \) find the probability that the population will survive forever.
(3) Assume \( X_0 = 1 \). What is the probability that the population will go extinct in the second generation, given that that is not extinct in the first?
(4) If \( p = 1/2 \), what is the probability that the population will go extinct in the third generation, given that is alive in the second generation?

Problem 5
(1) You need a computer for this. Do problem 2.12 from your book, page 60.

Problem 6
Let \( X_n, Y_n, Z_n \) be Markov Chains on the non-negative integers with transition probabilities given below. Decide whether these chains are transient, null recurrent or positive recurrent, and find the invariant distributions whenever is appropriate. For this problem, is impossible to use the recursion formula methods (why?)

(1) \( p_X(x, 0) = 1/(x + 2) \), \( p_X(x, x + 1) = (x + 1)/(x + 2) \).
(2) \( p_Y(x, 0) = (x + 1)/(x + 2) \), \( p_Y(x, x + 1) = 1/(x + 2) \).
(3) \( p_Z(x, 0) = 1/(x^2 + 2) \), \( p_Z(x, x + 1) = (x^2 + 1)/(x^2 + 2) \).

Problem 7
Let \( X_n \) denote the number of individuals alive at time \( n \) in a branching process, with the offspring distribution average \( \mu > 1 \). Assume further that \( X_0 = 1 \). Let \( \phi \) denote the probability generating function of the offspring distribution and let \( \alpha \) denote the extinction probability.

(1) Explain why \( \phi'(\alpha) < 1 \).
(2) Let \( a_n = \mathbb{P}\{X_n = 0\} \). Use part (1) to show that there exists a fixed number \( \rho \) so that for all \( n \) sufficiently large, \( a - a_{n+1} \leq \rho(a - a_n) \).
(3) Show that there exist \( b > 0, c < \infty \) so that for all \( n \)

\[
\mathbb{P}\{\text{eventual extinction}|X_n \neq 0\} \leq ce^{-bn}.
\]
(4) Explain why this implies that if a population will go extinct, it will most likely happen early.

**HINT:** Limit definition of derivative.