Roots & Factors

Roots of a polynomial

A root of a polynomial \( p(x) \) is a number \( \alpha \in \mathbb{R} \) such that \( p(\alpha) = 0 \).

Examples.

- 3 is a root of the polynomial \( p(x) = 2x - 6 \) because 
  \[ p(3) = 2(3) - 6 = 6 - 6 = 0 \]
- 1 is a root of the polynomial \( q(x) = 15x^2 - 7x - 8 \) since 
  \[ q(1) = 15(1)^2 - 7(1) - 8 = 15 - 7 - 8 = 0 \]
- \((\sqrt{2})^2 - 2 = 0\), so \(\sqrt{2}\) is a root of \(x^2 - 2\).

Be aware: What we call a root is what others call a “real root”, to emphasize that it is both a root and a real number. Since the only numbers we will consider in this course are real numbers, clarifying that a root is a “real root” won’t be necessary.

Factors

A polynomial \( q(x) \) is a factor of the polynomial \( p(x) \) if there is a third polynomial \( g(x) \) such that \( p(x) = q(x)g(x) \).

Example. \(3x^3 - x^2 + 12x - 4 = (3x - 1)(x^2 + 4)\), so \(3x - 1\) is a factor of \(3x^3 - x^2 + 12x - 4\). The polynomial \(x^2 + 4\) is also a factor of \(3x^3 - x^2 + 12x - 4\).

Factors and division

If you divide a polynomial \( p(x) \) by another polynomial \( q(x) \), and there is no remainder, then \( q(x) \) is a factor of \( p(x) \). That’s because if there’s no remainder, then \( \frac{p(x)}{q(x)} \) is a polynomial, and \( p(x) = q(x)\left(\frac{p(x)}{q(x)}\right) \). That’s the definition of \( q(x) \) being a factor of \( p(x) \).

If \( \frac{p(x)}{q(x)} \) has a remainder, then \( q(x) \) is not a factor of \( p(x) \).

Example. In the previous chapter we saw that 
\[
\frac{6x^2 + 5x + 1}{3x + 1} = 2x + 1
\]
Multiplying the above equation by $3x + 1$ gives

$$6x^2 + 5x + 1 = (3x + 1)(2x + 1)$$

so $3x + 1$ is a factor of $6x^2 + 5x + 1$.

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**Most important examples of roots**

Notice that the number $\alpha$ is a root of the linear polynomial $x - \alpha$ since $\alpha - \alpha = 0$.

You have to be able to recognize these types of roots when you see them.

<table>
<thead>
<tr>
<th>polynomial</th>
<th>root</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x - 2$</td>
<td>2</td>
</tr>
<tr>
<td>$x - 3$</td>
<td>3</td>
</tr>
<tr>
<td>$x - (-2)$</td>
<td>-2</td>
</tr>
<tr>
<td>$x + 2$</td>
<td>-2</td>
</tr>
<tr>
<td>$x + 15$</td>
<td>-15</td>
</tr>
<tr>
<td>$x - \alpha$</td>
<td>$\alpha$</td>
</tr>
</tbody>
</table>

**Linear factors give roots**

Suppose there is some number $\alpha$ such that $x - \alpha$ is a factor of the polynomial $p(x)$. We’ll see that $\alpha$ must be a root of $p(x)$.

That $x - \alpha$ is a factor of $p(x)$ means there is a polynomial $g(x)$ such that

$$p(x) = (x - \alpha)g(x)$$

Then

$$p(\alpha) = (\alpha - \alpha)g(\alpha)$$
$$= 0 \cdot g(\alpha)$$
$$= 0$$
Notice that it didn’t matter what polynomial \( g(x) \) was, or what number \( g(\alpha) \) was; \( \alpha \) is a root of \( p(x) \).

\[
\begin{align*}
\text{If } x - \alpha \text{ is a factor of } p(x), \\
\text{then } \alpha \text{ is a root of } p(x).
\end{align*}
\]

Examples.
- 2 is a root of \( p(x) = (x - 2)(\pi^7x^{15} - 27x^{11} + \frac{3}{4}x^5 - x^3) \) because
  \[
p(2) = (2 - 2)(\pi^72^{15} - 27(2)^{11} + \frac{3}{4}2^5 - 2^3) \\
  = 0 \cdot (\pi^72^{15} - 27(2)^{11} + \frac{3}{4}2^5 - 2^3) \\
  = 0
\]
- 4 is a root of \( q(x) = (x - 4)(x^{101} - x^{57} - 17x^3 + x) \)
- \(-2, 1, \) and \(5\) are roots of the polynomial \( 3(x + 2)(x - 1)(x - 5) \).

Roots give linear factors

Suppose the number \( \alpha \) is a root of the polynomial \( p(x) \). That means that \( p(\alpha) = 0 \). We’ll see that \( x - \alpha \) must be a factor of \( p(x) \).

Let’s start by dividing \( p(x) \) by \( (x - \alpha) \). Remember that when you divide a polynomial by a linear polynomial, the remainder is always a constant. So we’ll get something that looks like

\[
\frac{p(x)}{(x - \alpha)} = g(x) + \frac{c}{(x - \alpha)}
\]

where \( g(x) \) is a polynomial and \( c \in \mathbb{R} \) is a constant.

Next we can multiply the previous equation by \( (x - \alpha) \) to get

\[
p(x) = (x - \alpha)\left(g(x) + \frac{c}{(x - \alpha)}\right) \\
= (x - \alpha)g(x) + (x - \alpha)\frac{c}{(x - \alpha)} \\
= (x - \alpha)g(x) + c
\]
That means that
\[
p(\alpha) = (\alpha - \alpha)g(\alpha) + c = 0 \cdot g(\alpha) + c = 0 + c = c
\]
Now remember that \(p(\alpha) = 0\). We haven’t used that information in this problem yet, but we can now: because \(p(\alpha) = 0\) and \(p(\alpha) = c\), it must be that \(c = 0\). Therefore,
\[
p(x) = (x - \alpha)g(x) + c = (x - \alpha)g(x)
\]
That means that \(x - \alpha\) is a factor of \(p(x)\), which is what we wanted to check.

\[\text{If } \alpha \text{ is a root of } p(x), \text{ then } x - \alpha \text{ is a factor of } p(x)\]

**Example.** It’s easy to see that 1 is a root of \(p(x) = x^3 - 1\). Therefore, we know that \(x - 1\) is a factor of \(p(x)\). That means that \(p(x) = (x - 1)g(x)\) for some polynomial \(g(x)\).

To find \(g(x)\), divide \(p(x)\) by \(x - 1\):
\[
g(x) = \frac{p(x)}{x - 1} = \frac{x^3 - 1}{x - 1} = x^2 + x + 1
\]
Hence, \(x^3 - 1 = (x - 1)(x^2 + x + 1)\).

We were able to find two factors of \(x^3 - 1\) because we spotted that the number 1 was a root of \(x^3 - 1\).

\[
* * * * * * * * * * * * *
\]

**Roots and graphs**

If you put a root into a polynomial, 0 comes out. That means that if \(\alpha\) is a root of \(p(x)\), then \((\alpha, 0) \in \mathbb{R}^2\) is a point in the graph of \(p(x)\). These points are exactly the \(x\)-intercepts of the graph of \(p(x)\).

\[\text{The roots of a polynomial are exactly the } x\text{-intercepts of its graph.}\]
Examples.

- Below is the graph of a polynomial $p(x)$. The graph intersects the $x$-axis at 2 and 4, so 2 and 4 must be roots of $p(x)$. That means that $(x - 2)$ and $(x - 4)$ are factors of $p(x)$.

- Below is the graph of a polynomial $q(x)$. The graph intersects the $x$-axis at $-3$, 2, and 5, so $-3$, 2, and 5 are roots of $q(x)$, and $(x + 3)$, $(x - 2)$, and $(x - 5)$ are factors of $q(x)$. 

Degree of a product is the sum of degrees of the factors

Let’s take a look at some products of polynomials that we saw before in the chapter on “Basics of Polynomials”:

The leading term of \((2x^2 - 5x)(-7x + 4)\) is \(-14x^3\). This is an example of a degree 2 and a degree 1 polynomial whose product equals 3. Notice that \(2 + 1 = 3\)

The product \(5(x - 2)(x + 3)(x^2 + 3x - 7)\) is a degree 4 polynomial because its leading term is \(5x^4\). The degrees of 5, \((x - 2)\), \((x + 3)\), and \((x^2 + 3x - 7)\) are 0, 1, 1, and 2, respectively. Notice that \(0 + 1 + 1 + 2 = 4\).

The degrees of \((2x^3 - 7)\), \((x^5 - 3x + 5)\), \((x - 1)\), and \((5x^7 + 6x - 9)\) are 3, 5, 1, and 7, respectively. The degree of their product,
\[
(2x^3 - 7)(x^5 - 3x + 5)(x - 1)(5x^7 + 6x - 9),
\]
equals 16 since its leading term is \(10x^{16}\). Once again, we have that the sum of the degrees of the factors equals the degree of the product: \(3 + 5 + 1 + 7 = 16\).

These three examples suggest a general pattern that always holds for factored polynomials (as long as the factored polynomial does not equal 0):

If a polynomial \(p(x)\) is factored into a product of polynomials, then the degree of \(p(x)\) equals the sum of the degrees of its factors.

Examples.

- The degree of \((4x^3 + 27x - 3)(3x^6 - 27x^3 + 15)\) equals \(3 + 6 = 9\).
- The degree of \(-7(x + 4)(x - 1)(x - 3)(x - 3)(x^2 + 1)\) equals \(0 + 1 + 1 + 1 + 1 + 2 = 6\).

Degree of a polynomial bounds the number of roots

Suppose \(p(x)\) is a polynomial that has \(n\) roots, and that \(p(x)\) is not the constant polynomial \(p(x) = 0\). Let’s name the roots of \(p(x)\) as \(\alpha_1, \alpha_2, \ldots, \alpha_n\).

Any root of \(p(x)\) gives a linear factor of \(p(x)\), so
\[
p(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)q(x)
\]
for some polynomial \(q(x)\).

Because the degree of a product is the sum of the degrees, the degree of \(p(x)\) is at least \(n\).
The degree of $p(x)$ (if $p(x) \neq 0$) is greater than or equal to the number of roots that $p(x)$ has.

Examples.

- $5x^4 - 3x^3 + 2x - 17$ has at most 4 roots.
- $4x^{723} - 15x^{52} + 37x^{14} - 7$ has at most 723 roots.
- Aside from the constant polynomial $p(x) = 0$, if a function has a graph that has infinitely many $x$-intercepts, then the function cannot be a polynomial.

If it were a polynomial, its number of roots (or alternatively, its number of $x$-intercepts) would be bounded by the degree of the polynomial, and thus there would only be finitely many $x$-intercepts.

To illustrate, if you are familiar with the graphs of the functions $\sin(x)$ and $\cos(x)$, then you’ll recall that they each have infinitely many $x$-intercepts. Thus, they cannot be polynomials. (If you are unfamiliar with $\sin(x)$ and $\cos(x)$, then you can ignore this paragraph.)
Exercises

1.) Name two roots of the polynomial \((x - 1)(x - 2)\).

2.) Name two roots of the polynomial \(-(x + 7)(x - 3)(x^4 + x^3 + 2x^2 + x + 1)\).

3.) Name four roots of the polynomial \(-\frac{2}{5}(x + \frac{7}{3})(x + \frac{1}{2})(x - \frac{4}{3})(x - \frac{9}{7})(x^2 + 1)\).

   It will help with #4-6 to know that each of the polynomials from those problems has a root that equals either \(-1, 0,\) or \(1\). Remember that if \(\alpha\) is a root of \(p(x)\), then \(\frac{p(x)}{x - \alpha}\) is a polynomial and \(p(x) = (x - \alpha)\frac{p(x)}{x - \alpha}\).

4.) Write \(x^3 + 4x - 5\) as a product of a linear and a quadratic polynomial.

5.) Write \(x^3 + x\) as a product of a linear and a quadratic polynomial. (Hint: you could use the distributive law here.)

6.) Write \(x^5 + 3x^4 + x^3 - x^2 - x - 1\) as a product of a linear and a quartic polynomial.

7.) The graph of a polynomial \(p(x)\) is drawn below. Identify as many roots and factors of \(p(x)\) as you can.

8.) The graph of a polynomial \(q(x)\) is drawn below. Identify as many roots and factors of \(q(x)\) as you can.
For #9-13, determine the degree of the given polynomial.

9.) \((x + 3)(x - 2)\)
10.) \((3x + 5)(4x^2 + 2x - 3)\)
11.) \(-17(3x^2 + 20x - 4)\)
12.) \(4(x - 1)(x - 1)(x - 2)(x^2 + 7)(x^2 + 3x - 4)\)
13.) \(5(x - 3)(x^2 + 1)\)

14.) (True/False) \(7x^5 + 13x^4 - 3x^3 - 7x^2 + 2x - 1\) has 8 roots.

For #15-17, divide the polynomials. You can use synthetic division for #17 if you’d like.

15.) \[
\frac{x^6 - 2x^5 + 6x^4 - 10x^3 + 14x^2 - 10x + 14}{x^2 + 3}
\]
16.) \[
\frac{-2x^3 + x^2 + 4x - 6}{2x - 1}
\]
17.) \[
\frac{-2x^3 + 4x - 6}{x - 2}
\]