Piecewise Defined Functions

Most of the functions that we've looked at this semester can be expressed as a single equation. For example, \( f(x) = 3x^2 - 5x + 2 \), or \( g(x) = \sqrt{x - 1} \), or \( h(x) = e^{3x} - 1 \).

Sometimes an equation can’t be described by a single equation, and instead we have to describe it using a combination of equations. Such functions are called piecewise defined functions, and probably the easiest way to describe them is to look at a couple of examples.

First example. The function \( g : \mathbb{R} \to \mathbb{R} \) is defined by

\[
g(x) = \begin{cases} 
  x^2 - 1 & \text{if } x \in (-\infty, 0]; \\
  x - 1 & \text{if } x \in [0, 4]; \\
  3 & \text{if } x \in [4, \infty). 
\end{cases}
\]

The function \( g \) is a piecewise defined function. It is defined using three functions that we’re more comfortable with: \( x^2 - 1 \), \( x - 1 \), and the constant function \( 3 \). Each of these three functions is paired with an interval that appears on the right side of the same line as the function: \( (-\infty, 0] \), and \( [0, 4] \), and \( [4, \infty) \) respectively.

If you want to find \( g(x) \) for a specific number \( x \), first locate which of the three intervals that particular number \( x \) is in. Once you’ve decided on the correct interval, use the function that interval is paired with to determine \( g(x) \).

If you want to find \( g(2) \), first check that \( 2 \in [0, 4] \). Therefore, we should use the equation \( g(x) = x - 1 \), because \( x - 1 \) is the function that appears on the same line as the interval \( [0, 4] \). That means that \( g(2) = 2 - 1 = 1 \).

To find \( g(5) \), notice that \( 5 \in [4, \infty) \). That means we should be looking at the third interval used in the definition of \( g(x) \), and the function coupled with that interval is the constant function \( 3 \). Therefore, \( g(5) = 3 \).

Let’s look at one more number. Let’s find \( g(0) \). First we have to decide which of the three intervals used in the definition of \( g(x) \) contains the number \( 0 \). Notice that there’s some ambiguity here because \( 0 \) is contained in both the interval \( (-\infty, 0] \) and in the interval \( [0, 4] \). Whenever there’s ambiguity, choose either of the intervals that are options. Either of the functions that these intervals are paired with will give you the same result. That is, \( 0^2 - 1 = -1 \) is the same number as \( 0 - 1 = -1 \), so \( g(0) = -1 \).
To graph \( g(x) \), graph each of the pieces of \( g \). That is, graph \( g : (−∞, 0] \rightarrow \mathbb{R} \) where \( g(x) = x^2 − 1 \), and graph \( g : [0, 4] \rightarrow \mathbb{R} \) where \( g(x) = x − 1 \), and graph \( g : [4, \infty) \rightarrow \mathbb{R} \) where \( g(x) = 3 \). Together, these three pieces make up the graph of \( g(x) \).

Graph of \( g : (−∞, 0] \rightarrow \mathbb{R} \) where \( g(x) = x^2 − 1 \).

Graph of \( g : [0, 4] \rightarrow \mathbb{R} \) where \( g(x) = x − 1 \).

Graph of \( g : [4, \infty) \rightarrow \mathbb{R} \) where \( g(x) = 3 \).
To graph $g(x)$, draw the graphs of all three of its pieces.

Second example. The function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} (x - 3)^2 + 2 & \text{if } x \neq 3; \\ 4 & \text{if } x = 3. \end{cases}$$

This function is made up of two pieces. Either $x \neq 3$, in which case $f(x) = (x - 3)^2 + 2$. Or $x = 3$, and then $f(3) = 4$.

Graph of the first piece of $f(x)$: the graph of $x^2$ shifted right 3 and up 2 with the vertex removed.
(\text{Remember that a little circle means that point is not a point of the graph.})
Graph of the second piece of $f(x)$: a single giant dot.

Graph of both pieces, and hence the entire graph, of $f(x)$. 
Absolute value

The most important piecewise defined function in calculus is the absolute value function that is defined by

\[ |x| = \begin{cases} 
-x & \text{if } x \in (-\infty, 0]; \\
\phantom{-}x & \text{if } x \in [0, \infty). 
\end{cases} \]

The domain of the absolute value function is \( \mathbb{R} \). The range of the absolute value function is the set of non-negative numbers. The number \( |x| \) is called the absolute value of \( x \).

For examples of how this function works, notice that \( |4| = 4 \), \( |0| = 0 \), and \( |-3| = 3 \). If \( x \) is positive or 0, then the absolute value of \( x \) is \( x \) itself. If \( x \) is negative, then \( |x| \) is the positive number that you’d get from “erasing” the negative sign: \( |-10| = 10 \) and \( |-\frac{1}{2}| = \frac{1}{2} \).

Graph of the absolute value function.

Another interpretation of the absolute value function, and the one that’s most important for calculus, is that the absolute value of a number is the same as its distance from 0. That is, the distance between 0 and 5 is \( |5| = 5 \), the distance between 0 and -7 is \( |-7| = 7 \), and the distance between 0 and 0 is \( |0| = 0 \).
Let’s look at the graph of say $|x - 3|$. It’s the graph of $|x|$ shifted right by 3.

You might guess from the graph of $|x - 3|$, that $|x - 3|$ is the function that measures the distance between $x$ and 3, and that’s true. Similarly, $|x - 6|$ is the distance between $x$ and 6, $|x + 2|$ is the distance between $x$ and $-2$, and more generally, $|x - y|$ is the distance between $x$ and $y$.

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Solving inequalities involving absolute values

The inequality $|x| < 5$ means that the distance between $x$ and 0 is less than 5. Therefore, $x$ is between $-5$ and 5. Another way to write the previous sentence is $-5 < x < 5$. 
Notice in the above paragraph that the precise number 5 wasn’t really important for the problem. We could have replaced 5 with any positive number \(c\) to obtain the following translation.

\[
|x| < c \text{ means } -c < x < c
\]

For example, writing \(|x| < 2\) means the same as writing \(-2 < x < 2\), and \(|2x - 3| < \frac{1}{3}\) means the same as \(-\frac{1}{3} < 2x - 3 < \frac{1}{3}\).

We can use the above rule to help us solve some inequalities that involve absolute values.

**Problem.** Solve for \(x\) if \(|-3x + 4| < 2\).

**Solution.** We know from the explanation above that \(-2 < -3x + 4 < 2\).

Subtracting 4 from all three of the quantities in the previous inequality yields \(-2 - 4 < -3x < 2 - 4\), and that can be simplified as \(-6 < -3x < -2\).

Next divide by \(-3\), keeping in mind that dividing an inequality by a negative number “flips” the inequalities. The result will be \(\frac{2}{3} > x > \frac{2}{3}\), which can be simplified as \(2 > x > \frac{2}{3}\). That’s the answer.

The inequality \(2 > x > \frac{2}{3}\) could also be written as \(\frac{2}{3} < x < 2\), or as \(x \in (\frac{2}{3}, 2)\).
Problem. Solve for $x$ if $|2x - 1| < 3$.

Solution. Write the inequality from the problem as $-3 < 2x - 1 < 3$.

Add 1 to get $-2 < 2x < 4$, and divide by 2 to get $-1 < x < 2$.

Two important rules for absolute values

For the two rules below, $a, b, c \in \mathbb{R}$. Each rule is important for calculus. They’ll be explained in class.

1. $|ab| = |a||b|
2. $|a - c| \leq |a - b| + |b - c|$ (triangle inequality)
Exercises

1.) Suppose $f(x)$ is the piecewise defined function given by

$$f(x) = \begin{cases} 
  x + 1 & \text{if } x \in (-\infty, 2); \\
  x + 3 & \text{if } x \in [2, \infty).
\end{cases}$$

What is $f(0)$? What is $f(10)$? What is $f(2)$? Graph $f(x)$.

2.) Suppose $g(x)$ is the piecewise defined function given by

$$g(x) = \begin{cases} 
  3 & \text{if } x \in [1, 5]; \\
  1 & \text{if } x \in (5, \infty).
\end{cases}$$

What is $g(1)$? What is $g(100)$? What is $g(5)$? Graph $g(x)$.

3.) Suppose $h(x)$ is the piecewise defined function given by

$$h(x) = \begin{cases} 
  5 & \text{if } x \in (1, 3); \\
  x + 2 & \text{if } x \in [3, 8).
\end{cases}$$

What is $h(2)$? What is $h(7)$? What is $h(3)$? Graph $h(x)$.

4.) Suppose $f(x)$ is the piecewise defined function given by

$$f(x) = \begin{cases} 
  2 & \text{if } x \in [-3, 0); \\
  e^x & \text{if } x \in [0, 2]; \\
  3x - 2 & \text{if } x \in (2, \infty).
\end{cases}$$

What is $f(-2)$? What is $f(0)$? What is $f(2)$? What is $f(15)$? Graph $f(x)$.

5.) Suppose $g(x)$ is the piecewise defined function given by

$$g(x) = \begin{cases} 
  (x - 1)^2 & \text{if } x \in (-\infty, 1]; \\
  \log_e(x) & \text{if } x \in [1, 5]; \\
  \log_e(5) & \text{if } x \in [5, \infty).
\end{cases}$$

What is $g(0)$? What is $g(1)$? What is $g(5)$? What is $g(20)$? Graph $g(x)$. 

6.) Suppose $h(x)$ is the piecewise defined function given by

$$h(x) = \begin{cases} e^x & \text{if } x \neq 2; \\ 1 & \text{if } x = 2. \end{cases}$$

What is $h(0)$? What is $h(2)$? What is $h(\log_e(17))$? Graph $h(x)$.

7.) Write the following numbers as integers: $|8 - 5|$, $|-10 - 5|$, and $|5 - 5|$. The function $|x - 5|$ measures the distance between $x$ and which number?

8.) Write the following numbers as integers: $|1 - 2|$, $|3 - 2|$, and $|2 - 2|$. The function $|x - 2|$ measures the distance between $x$ and which number?

9.) Write the following numbers as integers: $|3 + 4|$, $|-1 + 4|$, $|-4 + 4|$. The function $|x + 4|$ measures the distance between $x$ and which number?

10.) The function $|x - y|$ measures the distance between $x$ and which number?

11.) Solve for $x$ if $|5x - 2| < 7$.

12.) Solve for $x$ if $|3x + 4| < 1$.

13.) Solve for $x$ if $|-2x + 3| < 5$. 
