Logarithms

If \( a > 1 \) or \( 0 < a < 1 \), then the exponential function \( f : \mathbb{R} \to (0, \infty) \) defined as \( f(x) = a^x \) is one-to-one and onto. That means it has an inverse function.

If either \( a > 1 \) or \( 0 < a < 1 \), then the inverse of the function \( a^x \) is

\[
\log_a : (0, \infty) \to \mathbb{R}
\]

and it’s called a logarithm of base \( a \).

That \( a^x \) and \( \log_a(x) \) are inverse functions means that

\[
a^{\log_a(x)} = x
\]

and

\[
\log_a(a^x) = x
\]

**Problem.** Find \( x \) if \( 2^x = 15 \).

**Solution.** The inverse of an exponential function with base 2 is \( \log_2 \). That means that we can erase the exponential base 2 from the left side of \( 2^x = 15 \) as long as we apply \( \log_2 \) to the right side of the equation. That would leave us with \( x = \log_2(15) \).

The final answer is \( x = \log_2(15) \). You stop there. \( \log_2(15) \) is a number. It is a perfectly good number, just like 5, -7, or \( \sqrt[3]{15} \) are. With some more experience, you will become comfortable with the fact that \( \log_2(15) \) cannot be simplified anymore than it already is, just like \( \sqrt[3]{15} \) cannot be simplified anymore than it already is. But they are both perfectly good numbers.

**Problem.** Solve for \( x \) where \( \log_4(x) = 3 \).

**Solution.** We can erase \( \log_4 \) from the left side of the equation by applying its inverse, exponential base 4, to the right side of the equation. That would give us \( x = 4^3 \). Now \( 4^3 \) can be simplified; it’s 64. So the final answer is \( x = 64 \).
**Problem.** Write \( \log_3(81) \) as an integer in standard form.

**Solution.** The trick to solving a problem like this is to rewrite the number being put into the logarithm — in this problem, 81 — as an exponential whose base is the same as the base of the logarithm — in this problem, the base is 3.

Being able to write 81 as an exponential in base 3 will either come from your comfort with exponentials, or from guess-and-check methods. Whether it’s immediately obvious to you or not, you can check that \( 81 = 3^4 \). (Notice that \( 3^4 \) is an exponential of base 3.) Therefore, \( \log_3(81) = \log_3(3^4) \).

Now we use that exponential base 3 and logarithm base 3 are inverse functions to see that \( \log_3(3^4) = 4 \).

To summarize this process in one line,

\[
\log_3(81) = \log_3(3^4) = 4
\]

**Problem.** Write \( \log_4(16) \) as an integer in standard form.

**Solution.** This is a logarithm of base 4, so we write 16 as an exponential of base 4: \( 16 = 4^2 \). Then,

\[
\log_4(16) = \log_4(4^2) = 2
\]
Graphing logarithms

Recall that if you know the graph of a function, you can find the graph of its inverse function by flipping the graph over the line $x = y$.

Below is the graph of a logarithm of base $a > 1$. Notice that the graph grows taller, but very slowly, as it moves to the right.

Below is the graph of a logarithm when the base is between 0 and 1.
**Two base examples**

If $a^x = y$, then $x = \log_a(y)$. Below are some examples in base 10.

<table>
<thead>
<tr>
<th>$10^x$</th>
<th>$\log_{10}(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-3} = \frac{1}{1,000}$</td>
<td>$-3 = \log_{10}(\frac{1}{1,000})$</td>
</tr>
<tr>
<td>$10^{-2} = \frac{1}{100}$</td>
<td>$-2 = \log_{10}(\frac{1}{100})$</td>
</tr>
<tr>
<td>$10^{-1} = \frac{1}{10}$</td>
<td>$-1 = \log_{10}(\frac{1}{10})$</td>
</tr>
<tr>
<td>$10^0 = 1$</td>
<td>$0 = \log_{10}(1)$</td>
</tr>
<tr>
<td>$10^1 = 10$</td>
<td>$1 = \log_{10}(10)$</td>
</tr>
<tr>
<td>$10^2 = 100$</td>
<td>$2 = \log_{10}(100)$</td>
</tr>
<tr>
<td>$10^3 = 1,000$</td>
<td>$3 = \log_{10}(1,000)$</td>
</tr>
<tr>
<td>$10^4 = 10,000$</td>
<td>$4 = \log_{10}(10,000)$</td>
</tr>
<tr>
<td>$10^5 = 100,000$</td>
<td>$5 = \log_{10}(100,000)$</td>
</tr>
</tbody>
</table>
Below are the graphs of the functions $10^x$ and $\log_{10}(x)$. The graphs are another way to display the information from the previous chart.
This chart contains examples of exponentials and logarithms in base 2.

<table>
<thead>
<tr>
<th>$2^x$</th>
<th>$\log_2(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^{-4} = \frac{1}{16}$</td>
<td>$-4 = \log_2\left(\frac{1}{16}\right)$</td>
</tr>
<tr>
<td>$2^{-3} = \frac{1}{8}$</td>
<td>$-3 = \log_2\left(\frac{1}{8}\right)$</td>
</tr>
<tr>
<td>$2^{-2} = \frac{1}{4}$</td>
<td>$-2 = \log_2\left(\frac{1}{4}\right)$</td>
</tr>
<tr>
<td>$2^{-1} = \frac{1}{2}$</td>
<td>$-1 = \log_2\left(\frac{1}{2}\right)$</td>
</tr>
<tr>
<td>$2^0 = 1$</td>
<td>$0 = \log_2(1)$</td>
</tr>
<tr>
<td>$2^1 = 2$</td>
<td>$1 = \log_2(2)$</td>
</tr>
<tr>
<td>$2^2 = 4$</td>
<td>$2 = \log_2(4)$</td>
</tr>
<tr>
<td>$2^3 = 8$</td>
<td>$3 = \log_2(8)$</td>
</tr>
<tr>
<td>$2^4 = 16$</td>
<td>$4 = \log_2(16)$</td>
</tr>
<tr>
<td>$2^5 = 32$</td>
<td>$5 = \log_2(32)$</td>
</tr>
<tr>
<td>$2^6 = 64$</td>
<td>$6 = \log_2(64)$</td>
</tr>
</tbody>
</table>
The information from the previous page is used to draw the graphs of $2^x$ and $\log_2(x)$.
Rules for logarithms

The most important rule for exponential functions is \( a^x a^y = a^{x+y} \). Because \( \log_a(x) \) is the inverse of \( a^x \), it satisfies the “opposite” of this rule:

\[
\log_a(z) + \log_a(w) = \log_a(zw)
\]

Here’s why the above equation is true:

\[
\log_a(z) + \log_a(w) = \log_a(a^{\log_a(z)}a^{\log_a(w)}) \\
= \log_a(a^{\log_a(z)+\log_a(w)}) \\
= \log_a(zw)
\]

The next two rules are different versions of the rule above:

\[
\log_a(z) - \log_a(w) = \log_a\left(\frac{z}{w}\right)
\]

\[
\log_a(z^w) = w \log_a(z)
\]

Because \( a^0 = 1 \), it’s also true that

\[
\log_a(1) = 0
\]
Change of base formula

Let’s say that you wanted to know a decimal number that is close to \( \log_3(7) \), and you have a calculator that can only compute logarithms in base 10. Your calculator can still help you with \( \log_3(7) \) because the change of base formula tells us how to use logarithms in one base to compute logarithms in another base.

The change of base formula is:

\[
\log_a(x) = \frac{\log_b(x)}{\log_b(a)}
\]

In our example, you could use your calculator to find that 0.845 is a decimal number that is close to \( \log_{10}(7) \), and that 0.477 is a decimal number that is close to \( \log_{10}(3) \). Then according to the change of base formula

\[
\log_3(7) = \frac{\log_{10}(7)}{\log_{10}(3)}
\]

is close to the decimal number

\[
\begin{align*}
0.845 \\
0.477
\end{align*}
\]

which itself is close to 1.771.

We can see why the change of base formula is true. First notice that

\[
\log_a(x) \log_b(a) = \log_b(a^{\log_a(x)}) = \log_b(x)
\]

The first equal sign above uses the third rule from the section on rules for logarithms. The second equal sign uses that \( a^x \) and \( \log_a(x) \) are inverse functions.

Now divide the equation above by \( \log_b(a) \), and we’re left with the change of base formula.

Base confusion

To a mathematician, \( \log(x) \) means \( \log_e(x) \). Most calculators use \( \log(x) \) to mean \( \log_{10}(x) \). Sometimes in computer science, \( \log(x) \) means \( \log_2(x) \). A lot of people use \( \ln(x) \) to mean \( \log_e(x) \). (\( \ln(x) \) is called the “natural logarithm”.)

In this class, we’ll never write the expression \( \log(x) \) or \( \ln(x) \). We’ll always be explicit with our bases and write logarithms of base 10 as \( \log_{10}(x) \), logarithms of base 2 as \( \log_2(x) \), and logarithms of base \( e \) as \( \log_e(x) \). To be safe, when
doing math in the future, always ask what base a logarithm is if it’s not clear to you.

* * * * * * * * * * * *
Exercises

For #1-8, match each of the numbered functions on the left with the lettered function on the right that is its inverse.

1.) \( x + 7 \) . A.) \( x^7 \)

2.) \( 3x \) B.) \( \frac{x}{3} \)

3.) \( \sqrt{x} \) C.) \( 3^x \)

4.) \( 7^x \) D.) \( \sqrt[3]{x} \)

5.) \( \frac{x}{7} \) E.) \( x - 7 \)

6.) \( \log_3(x) \) F.) \( x + 3 \)

7.) \( x - 3 \) G.) \( 7x \)

8.) \( x^3 \) H.) \( \log_7(x) \)

Graph the functions in #9-12.

9.) \( \log_{10}(x - 3) \)

10.) \( \log_2(x + 5) \)

11.) \( \log_{\frac{1}{3}}(x) + 4 \)

12.) \(-3 \log_e(x)\)
For #13-21, write the given number as a rational number in standard form, for example, 2, \(-3\), \(\frac{3}{4}\), and \(-\frac{1}{5}\) are rational numbers in standard form. These are the exact same questions, in the same order, as those from #16-24 in the chapter on Exponential Functions. They’re just written in the language of logarithms instead.

13.) \(\log_4(16)\)

14.) \(\log_2(8)\)

15.) \(\log_{10}(10,000)\)

16.) \(\log_3(9)\)

17.) \(\log_5(125)\)

18.) \(\log_{\frac{1}{2}}(16)\)

19.) \(\log_{\frac{1}{4}}(64)\)

20.) \(\log_8(\frac{1}{4})\)

21.) \(\log_{27}(\frac{1}{9})\)

For #22-29, decide which is the greatest integer that is less than the given number. For example, if you’re given the number \(\log_2(9)\) then the answer would be 3. You can see that this is the answer by marking 9 on the \(x\)-axis of the graph of \(\log_2(x)\) that’s drawn earlier in this chapter. You can use the graph and the point you marked to see that \(\log_2(9)\) is between 3 and 4, so 3 is the greatest of all of the integers that are less than (or below) \(\log_2(9)\).

22.) \(\log_{10}(15)\)

23.) \(\log_{10}(950)\)

24.) \(\log_2(50)\)

25.) \(\log_2(3)\)

26.) \(\log_3(18)\)

27.) \(\log_{10}(\frac{1}{19})\)
28.) \( \log_2\left(\frac{1}{10}\right) \)
29.) \( \log_3\left(\frac{1}{10}\right) \)

In the remaining exercises, use that \( \log_a(x) \) and \( a^x \) are inverse functions to solve for \( x \).

30.) \( \log_4(x) = -2 \)
31.) \( \log_6(x) = 2 \)
32.) \( \log_3(x) = -3 \)
33.) \( \log_{\frac{1}{10}}(x) = -5 \)
34.) \( e^x = 17 \)
35.) \( e^x = 53 \)
36.) \( \log_e(x) = 5 \)
37.) \( \log_e(x) = -\frac{1}{3} \)