# The 3D Stress Field at the Edge of a Circular Hole and the Interface of a Laminated Composite Plate

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**ABSTRACT:** The author in this paper investigates the 3D stress field in the immediate vicinity of a bonded interface and the free edge of a hole in a laminated composite plate. The laminates are assumed to be of homogeneous and isotropic materials, but of different elastic properties. As to loading, a uniform tensile load is applied in the plane of the plate and at points far remote from the hole (see Figure 1).

In constructing the local asymptotic solution, the author assumes the 3D field in a certain form which then permits a straightforward Williams approach for the determination of the stress singularities. The displacement and stress fields are recovered explicitly and a stress singularity is shown to exist for certain shear moduli ratios of $G2/G1$. In general, the stress singularity is shown to be a function of the respective ratios of the shear moduli and Poisson's. Moreover, the presence of a second singularity is observed which has significant implications to the problem of adhesion. An extension of the results to anisotropic layers is also discussed.

## INTRODUCTION

COMPOSITE LAMINATES ARE being used extensively in aerospace structures. Many of these laminates, however, develop high interlaminar stresses near holes that ultimately cause delaminations. Knowledge of the stress field in such areas is of great importance to the designer. Unfortunately, the problem is 3D in nature and as a result it is very difficult to solve. Moreover, the presence of two different material interfaces makes the problem even more complex.

Problems of this type have been investigated from a 2D point of view by many researchers and the results can be found in the literature. For example, in 1927 Knein [1] looked into the plane strain problem of an orthogonal elastic wedge bonded to a rigid base. In 1955, Rongved [2] investigated the problem of two bonded elastic half-spaces subjected to a concentrated force in the interior. Subsequently, in 1959, Williams [3] studied the stress field around a fault or a crack

in dissimilar media. The work was then generalized in 1965 by Rice and Sih [4] to also include arbitrary angles.

It was not until 1968 that Bogy [5] considered the general problem of two bonded quarter-planes of dissimilar isotropic, elastic, materials subjected to arbitrary boundary tractions. The problem was solved by a straightforward application of the Mellin transform in conjunction with the Airy stress function. In June of 1971, the same author [6] extended his work to also include dissimilar wedges of arbitrary angles. A few months later, Hein and Erdogan [7], using the same method of solution, independently reproduced the results by Bogy. Finally, in 1975 Westmann [8] studied the case of a wedge of an arbitrary angle which was bonded along a finite length to a half space. His analysis showed the presence of two singularities close to each other. Thus, elimination of the first singular term does not lead to a bounded stress field since the second singularity is still present.

Based on 3D considerations,* in 1979 Luk and Keer [9] investigated the stress field in an elastic half-space containing a partially embedded axially-loaded, rigid cylindrical rod. The problem was formulated in terms of Hankel integral transforms that was finally casted into a system of coupled singular integral equations the solution of which was sought numerically. The authors were able, however, to extract in the limit from the integral equations the characteristic equation governing the singular behavior at the intersection of the free surface and that of the rigid inclusion. Their result was in agreement with that obtained by Williams [10] for a right-angle corner with fixed-free boundary conditions.

In 1980 Haritos and Keer [11] investigated the stress field in a half-space containing an embedded rigid block under the conditions of plane strain. The problem was formulated by cleverly superimposing the solutions to the problems of horizontal and vertical line inclusions beneath an elastic half-space. By isolating the pertinent terms, the authors were able to extract directly from the integral equations the order of the stress singularity at both corners. Both results are in agreement with the Williams solution. Moreover, the authors point out the importance of the second singularity to the results of the load transfer problems.

In 1984, Delale [12] extended the solution reported by Alblas [13] for the equilibrium of one linear elastic layer with a hole to the case of two layers of different materials. He also looked into the stress singularity at the vicinity of the hole but the estimate of the singularity is based on 2D considerations. Finally, in 1986 Folias [14], utilizing the form of a general 3D solution for the equilibrium of linear elastic layers, which he developed in 1975 [15], was able to derive explicitly the 3D displacement and stress fields at the intersection of a hole and a free surface. The analysis revealed that the stresses at the corner are proportional to $\varrho^{\alpha-2}$ where $\varrho$ represents the local radius from the corner and $\alpha = 3.73959 \pm i\,1.11902$. It is interesting to note that the root is precisely the same as that obtained by Williams in his classic paper [10] for a $90°$ material corner with free-free stress boundaries.

The same general 3D solution can now be used to solve the exact 3D stress field in the vicinity of the intersection of an interface with the free surface of the

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*Due to the symmetry of the applied load, the problem is 2D.

hole in a laminated composite plate. Although our ultimate goal is the determination of the stress field in transversely isotropic laminae, in this paper we will investigate the simpler case where the laminae are made of homogeneous and isotropic materials.

## FORMULATION OF THE PROBLEM

Consider the equilibrium of a laminated composite plate which occupies the space $|x| < \infty$, $|y| < \infty$ and $|z| \le h$ and contains a cylindrical hole of radius $a$ whose generators are perpendicular to the bounding planes, namely $z = \pm h$. The plate is composed of laminates that are made of homogeneous and isotropic material but of different elastic properties. The laminates are assumed to be perfectly bonded at the interface. The plate is subjected to a uniform axisymmetric tensile strain field, applied far away from the hole (see Figure 1).

In the absence of body forces, the coupled differential equations governing the displacement functions $u_i^{(m)}$ are:

$$\frac{1}{1 - 2\nu_m} \frac{\partial e^{(m)}}{\partial x_i} + \nabla^2 u_i^{(m)} = 0; \qquad i = 1,2,3 \qquad m = 1,2 \qquad (1)$$

where $\nabla^2$ is the Laplacian operator, $\nu_m$ is Poisson's ratio, $u_i^{(1)}$ and $u_i^{(2)}$ represent the displacement functions in media 1 and 2 respectively, and
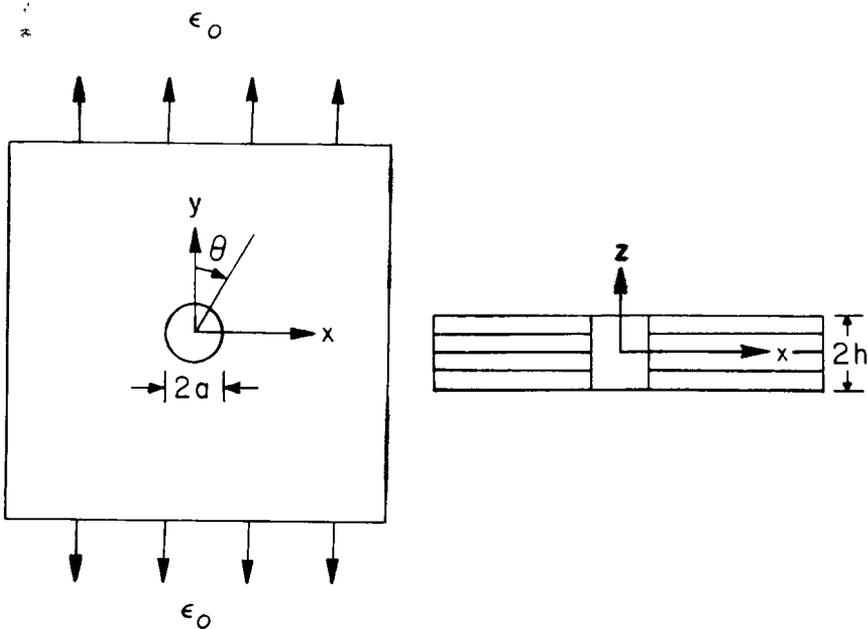


**Figure 1.** *Laminated plate weakened by a cylindrical hole.*

$$e^{(m)} \equiv \frac{\partial u_i^{(m)}}{\partial x_i} \quad i = 1,2,3; \quad m = 1,2 \tag{2}$$

The stress-displacement relations are given by Hooke's law as

$$\sigma_{ij}^{(m)} = \lambda_m e_{kk}^{(m)} \delta_{ij} + 2G e_{ij}^{(m)} \tag{3}$$

where $\lambda_m$ and $G_m$ are material constants describing media 1 and 2.

## METHOD OF SOLUTION

The main objective of this analysis is to derive an asymptotic solution that is valid in the immediate vicinity of the corner point, i.e. the point where the interface meets the free surface of the hole. Guided by a general solution to the equilibrium of linear elastic layers which the author constructed in a previous paper [15], we assume the complementary displacement field in the form:

$$u^{(m)} = \frac{1}{1 - 2\nu_m} \frac{\partial}{\partial x} \left\{ 2(1 - \nu_m) f_2^{(m)} + \frac{\partial f_1^{(m)}}{\partial z} + z \frac{\partial f_2^{(m)}}{\partial z} \right\} + \frac{\partial f_3^{(m)}}{\partial y} \tag{4}$$

$$v^{(m)} = \frac{1}{1 - 2\nu_m} \frac{\partial}{\partial y} \left\{ 2(1 - \nu_m) f_2^{(m)} + \frac{\partial f_1^{(m)}}{\partial z} + z \frac{\partial f_2^{(m)}}{\partial z} \right\} - \frac{\partial f_3^{(m)}}{\partial x} \tag{5}$$

$$w^{(m)} = \frac{1}{1 - 2\nu_m} \frac{\partial}{\partial z} \left\{ -2(1 - \nu_m) f_2^{(m)} + \frac{\partial f_1^{(m)}}{\partial z} + z \frac{\partial f_2^{(m)}}{\partial z} \right\} \tag{6}$$

where the functions $f_j^{(m)}, j = 1,2,3$, are three dimensional harmonic functions. If we furthermore assume that:

$$f_j^{(m)} = r^{-1/2} H_j^{(m)} (r - a, z) e^{i2\theta}; \quad j = 1,2,3 \tag{7}$$

then the functions $H_j^{(m)}$ must satisfy the following equation:

$$\frac{\partial^2 H_j^{(m)}}{\partial (r - a)^2} + \frac{\partial^2 H_j^{(m)}}{\partial z^2} - \frac{15}{4(a + r - a)^2} H_j^{(m)} = 0 \tag{8}$$

FRONT SURFACE

INTERFACE

① 

② 

BACK SURFACE

a
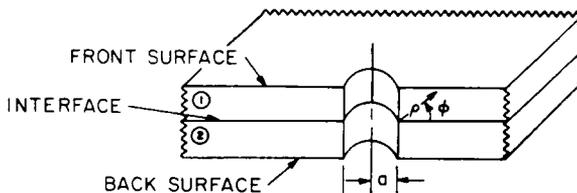
**Figure 2.** Definition of local coordinates at the corner.

It is found convenient at this stage to introduce the local coordinate system (see Figure 2)

$$r - a = \varrho \cos \phi$$

$$z = \varrho \sin \phi$$

in view of which, Equation (8) may now be written as

$$\frac{\partial^2 H_j^{(m)}}{\partial \varrho^2} + \frac{1}{\varrho} \frac{\partial H_j^{(m)}}{\partial \varrho} + \frac{1}{\varrho^2} \frac{\partial^2 H_j^{(m)}}{\partial \phi^2} - \frac{15 H_j^{(m)}}{4a^2[1 + (\varrho/a) \cos \phi]^2} = 0 \quad (9)$$

Under the assumption that the radius of the inclusion is sufficiently large, so that the condition $\varrho \ll a$ be meaningful, we seek the solution to Equation (9) in the form

$$H_j^{(m)} \sum_{n=0}^{\infty} \varrho^{\alpha+n} F_{jn}^{(m)} (\phi) \quad (10)$$

where $\alpha$ stands for a constant. Without going into the mathematical details, we construct the following series representations in ascending powers of $\varrho$:

$$\frac{2}{1 - 2\nu_m} \left\{ \frac{\partial f_1^{(m)}}{\partial z} + z \frac{\partial f_2^{(m)}}{\partial z} \right\} = \varrho^\alpha \{ [A_1^{(m)} + A_2^{(m)}] \sin (\alpha\phi)$$

$$+ [B_1^{(m)} + B_2^{(m)}] \cos (\alpha\phi) + 2C^{(m)} \sin (\alpha - 2)\phi \quad (11)$$

$$+ 2D^{(m)} \cos (\alpha - 2)\phi \} \cos (2\theta)$$

$$\frac{4(1 - \nu_m)}{1 - 2\nu_m} f_2^{(m)} = -\varrho^\alpha \{ [A_1^{(m)} - A_2^{(m)}] \sin (\alpha\phi)$$

$$\quad (12)$$

$$+ [B_1^{(m)} - B_2^{(m)}] \cos (\alpha\phi) \} \cos (2\theta)$$

$$f_3^{(m)} = \varrho^\alpha \{ A_3^{(m)} \sin (\alpha\phi) + B_3^{(m)} \cos (\alpha\phi) \} \sin (2\theta) \quad (13)$$

where

$$8(1 - \nu_m)C^{(m)} = \alpha[A_1^{(m)} - A_2^{(m)}] \quad (14)$$

and

$$8(1 - \nu_m)D^{(m)} = \alpha[B_1^{(m)} - B_2^{(m)}] \quad (15)$$

and $\alpha$, $A_j^{(m)}$, $B_j^{(m)}$, are constants to be determined from the boundary conditions. In particular,

$$\text{at } \phi = 0: \qquad \sigma_{zz}^{(1)} = \sigma_{zz}^{(2)} \tag{16}$$

$$\tau_{xz}^{(1)} = \tau_{xz}^{(2)} \tag{17}$$

$$\tau_{yz}^{(1)} = \tau_{yz}^{(2)} \tag{18}$$

$$u^{(1)} = u^{(2)} \tag{19}$$

$$v^{(1)} = v^{(2)} \tag{20}$$

$$w^{(1)} = w^{(2)} \tag{21}$$

$$\text{at } \phi = \pi/2: \qquad \tau_{rz}^{(1)} = 0 \tag{22}$$

$$\tau_{r\theta}^{(1)} = 0 \tag{23}$$

$$\sigma_{rr}^{(1)} = 0 \tag{24}$$

$$\text{at } \phi = -\pi/2: \qquad \tau_{rz}^{(2)} = 0 \tag{25}$$

$$\tau_{r\theta}^{(2)} = 0 \tag{26}$$

$$\sigma_{rr}^{(2)} = 0 \tag{27}$$

Substituting Equations (11)–(13) into Equations (16)–(27) one finds that all the boundary conditions are satisfied if one considers the following combinations to vanish:

$$\alpha B_1^{(1)} + \left[ \alpha + 8\nu_1 \left( \frac{1 - \nu_1}{1 - 2\nu_2} \right) - 4 \left( \frac{1 - \nu_1}{1 - 2\nu_1} \right) \right] D^{(1)} - (G_2/G_1) \left\{ \alpha B_1^{(2)} \right. \tag{28}$$

$$+ \left[ \alpha + 8\nu_2 \left( \frac{1 - \nu_2}{1 - 2\nu_2} \right) - 4 \left( \frac{1 - \nu_2}{1 - 2\nu_2} \right) \right] D^{(2)} \right\} = 0$$

$$\alpha [A_1^{(1)} + A_2^{(1)}] + 2(\alpha - 2)C^{(1)} - \left( \frac{G_2}{G_1} \right) \tag{29}$$

$$\{ \alpha [A_1^{(2)} + A_2^{(2)}] + 2(\alpha - 2)C^{(2)} \} = 0$$

$$A_3^{(1)} = A_3^{(2)} = 0 \tag{30}$$

$$\{ [A_1^{(1)} + A_2^{(1)}]\alpha + 2(\alpha - 2)C^{(1)} \} \tag{31}$$

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$$- \{[B_1^{(1)} + B_2^{(1)}]\alpha - 2(\alpha - 2)D^{(1)}\} \tan\left(\frac{\alpha\pi}{2}\right) = 0$$

$$B_3^{(1)} = 0 \tag{32}$$

$$\left\{\alpha\left(\frac{1-\nu_1}{1-2\nu_1}\right)[A_1^{(1)} - A_2^{(1)}] - A_1^{(1)} + \left[\alpha - 4\frac{1-\nu_1}{1-2\nu_1}\right]C^{(1)}\right\} \tan\left(\frac{\alpha\pi}{2}\right)$$

$$+ \left\{\alpha\left(\frac{1-\nu_1}{1-2\nu_1}\right)[B_1^{(1)} - B_2^{(1)}] - B_1^{(1)} + \left[\alpha - 4\frac{1-\nu_1}{1-2\nu_1}\right]D^{(1)}\right\} = 0 \tag{33}$$

$$[A_1^{(2)} + A_2^{(2)}]\alpha + [B_1^{(2)} + B_2^{(2)}]\alpha \tan\left(\frac{\alpha\pi}{2}\right)$$

$$- 2C^{(2)}(\alpha - 2) - 2(\alpha - 2)D^{(2)} \tan\left(\frac{\alpha\pi}{2}\right) = 0 \tag{34}$$

$$B_3^{(2)} = 0 \tag{35}$$

$$- \left\{\alpha\left(\frac{1-\nu_2}{1-2\nu_2}\right)[A_1^{(2)} - A_2^{(2)}] - A_1^{(2)} + \left[\alpha - 4\left(\frac{1-\nu_2}{1-2\nu_2}\right)\right]C^{(2)}\right\}$$

$$\tan\left(\frac{\alpha\pi}{2}\right) + \left\{\alpha\left(\frac{1-\nu_2}{1-2\nu_2}\right)[B_1^{(2)} - B_2^{(2)}] - B_1^{(2)}\right. \tag{36}$$

$$\left. + \left[\alpha - 4\frac{1-\nu_2}{1-2\nu_2}\right]D^{(2)}\right\} = 0$$

The characteristic value $\alpha$ may now be determined by setting the determinant of the algebraic system (28)–(36) equal to zero. Once the roots have been determined, the complete displacement and stress fields can be constructed in ascending powers of $\varrho$.

Without going into the mathematical details, the characteristic values $\alpha$ are constant and can easily be determined with the aid of a computer. Although the equation has an infinite number of complex roots, only the one root with a $1 < \min \mathrm{Re}\,\alpha < 2$ is relevant. In general, the characteristic values of $\alpha$ depend on the material properties of the laminae. Thus, the displacement field now reads:

**The Displacement Field**

$$u^{(m)} = \alpha\varrho^{\alpha-1}\{A_2^{(m)} \sin(\alpha - 1)\phi + B_2^{(m)} \cos(\alpha - 1)\phi$$

$$+ C^{(m)} \sin(\alpha - 3)\phi + D^{(m)} \cos(\alpha - 3)\phi\} \sin\theta \cos(2\theta) \tag{37}$$

$$+ 2\varrho^{\alpha-1}\{C^{(m)} \cos (\alpha - 2)\phi - D^{(m)} \sin (\alpha - 2)\phi\} \sin \phi \sin \theta \cos (2\theta)$$

$$v^{(m)} = \alpha\varrho^{\alpha-1}\{A_2^{(m)} \sin (\alpha - 1)\phi + B_2^{(m)} \cos (\alpha - 1)\phi$$

$$+ C^{(m)} \sin (\alpha - 3)\phi + D^{(m)} \cos (\alpha - 3)\phi\} \cos \theta \cos (2\theta) \tag{38}$$

$$+ 2\varrho^{\alpha-1}\{C_2^{(m)} \cos (\alpha - 2)\phi - D_2^{(m)} \sin (\alpha - 2)\phi\} \sin \phi \cos \theta \cos (2\theta)$$

$$w^{(m)} = \alpha\varrho^{\alpha-1}\{A_1^{(m)} \cos (\alpha - 1)\phi - B_1^{(m)} \sin (\alpha - 1)\phi$$

$$+ C^{(m)} \cos (\alpha - 3)\phi - D^{(m)} \sin (\alpha - 3)\phi\} \cos (2\theta) \tag{39}$$

$$- 2\varrho^{\alpha-1}\{C^{(m)} \cos (\alpha - 2)\phi - D^{(m)} \sin (\alpha - 2)\phi\} \cos \phi \cos (2\theta)$$

The analysis clearly shows that, in the neighborhood of the interface and the free surface of the hole, the stress field is proportional to $\varrho^{\alpha-2}$ and that for certain material properties it is singular. Moreover, the characteristic values $\alpha$ are independent of $\theta$ but they do depend on the material properties of the adjacent laminae. The first root $\alpha$ is found to be precisely the same as that of the corresponding 2D case reported by Bogy [6]. Figures 3 and 4 depict typical results for various material properties. Finally, in the limit as $G_2 \to \infty$, i.e., for a perfectly rigid laminae, the characteristic value of $\alpha = 1.7112$ is obtained. Moreover, as
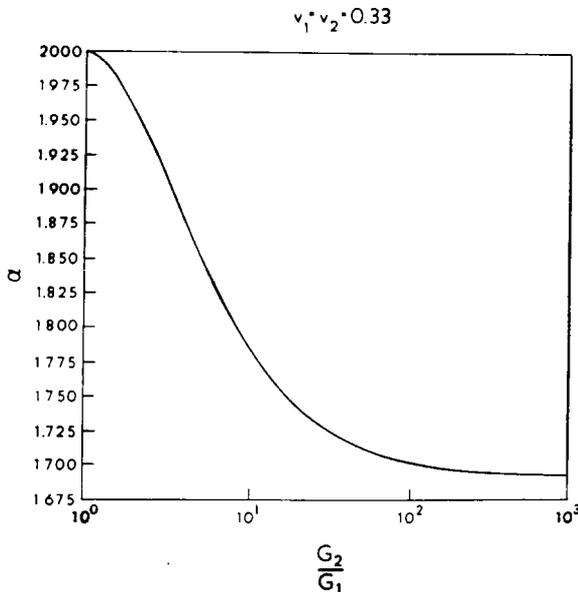


**Figure 3.** *Strength of the singularity vs. $G_2/G$, for $\nu_1 = \nu_2 = 0.33$.*
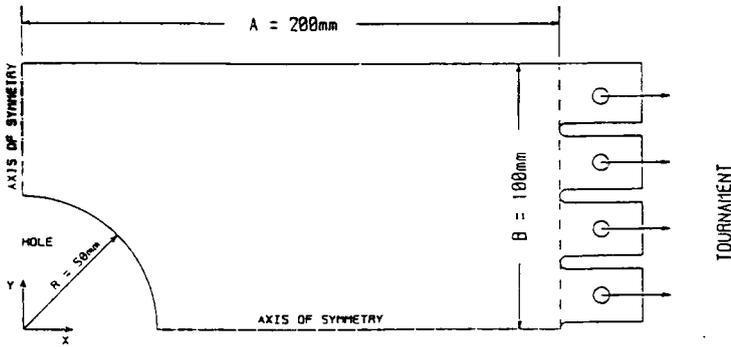
**Figure 4.** Strength of the singularity vs. $G_2/G_1$ for $\nu_1 = 0.33$, $\nu_2 = 0.25$.

$G_2 \to G_1$ and $\nu_2 \to \nu_1$, the solution of a continuous plate is recovered, a result that clearly meets our expectations.

An extension of this analysis to other angles of intersection with the free surface reveals the same results as those predicted by Bogy [6] for the case of plane strain. It may also be noted that the analysis confirms the presence of another weaker singularity which was first pointed out by Westmann [8]. While it is true that the singular stress field is dominated by the largest singularity, the presence of two singular terms has important implications to the problem of adhesion.

As a practical matter, the designer may now appropriately choose the material constants of the laminae so that the coefficients of the singular terms of the interfacial stresses vanish. There exists a sufficient number of constants that such an objective can indeed be accomplished.

Focusing next on the interface plane $z = 0$, we compute the shear and normal stresses, i.e.

$$\tau_{r\theta}^{(m)} = 0 + \ldots \tag{40}$$

$$\tau_{rz}^{(m)} = 2(\alpha - 1)G_m\varrho^{\alpha-2}\{\alpha A_1^{(m)} + [(\alpha - 2) \\ - 4(1 - \nu_m)]C^{(m)}\} \cos(2\theta) + \ldots \tag{41}$$

$$\sigma_{zz}^{(m)} = \frac{2G_m\nu_m}{(1 - 2\nu_m)}(\alpha - 1)\varrho^{\alpha-2}\left\{\alpha\left[B_2^{(m)} - \left(\frac{1 - \nu_m}{\nu_m}\right)B_1^{(m)}\right] \\ + \left[\alpha - \left(\frac{1 - \nu_m}{\nu_m}\right)(\alpha - 4)\right]D^{(m)}\right\}\cos(2\theta) + \ldots, \tag{42}$$

which indicates that only the shear stress $\tau_{rz}^{(m)}$ and the normal stress $\sigma_{zz}^{(m)}$ are of concern to the designer for they may very well contribute to possible debonding.

Perhaps it is appropriate here to note that in the case of transversely isotropic laminae the exponent $\alpha$ is no longer independent of $\theta$. In fact, it is a rather com-

plicated function involving the material properties of the laminae as well as an angular distribution in $\theta$. The results have recently been completed and will be reported in a subsequent paper.

In closing, it is noteworthy to point out the importance of the general 3D solution [15] for it reveals the inherent form of the solution at such neighborhoods and permits a simplistic analytical approach, a la Williams [10], for the determination of the stress singularities.

## ACKNOWLEDGEMENT

## REFERENCES

1. Knein, M. "Zur Theorie der Druckversuchs," *Abhandlungen der Aerodynamische Inst. u.d. Technische Hochschule, Aachen, Germany*, Vol. 7, pp. 43-62 (1927).

2. Rongved, L. "Force Interior to One of Two Joined Semi-Infinite Solids," *Second Midwestern Conference on Solid Mechanics*, pp. 1-13 (September 1955).

3. Williams, M. L. "The Stresses around a Fault or Crack in Dissimilar Media," *Bulletin of the Seismological Society of America, Vol. 49*, p. 199 (1959).

4. Rice, J. R. and G. C. Sih. "Plane Problems of Cracks in Dissimilar Media," *Journal of Applied Mechanics*, Vol. 32, *Trans. ASME* Vol. 87, Series E, p. 418 (1965).

5. Bogy, D. B. "Edge-Bonded Dissimilar Orthogonal Elastic Wedges under Normal and Shear Loading," *Journal of Appl. Mech.*, Vol. 35, *Trans. ASME*, pp. 460-466 (1968).

6. Bogy, D. B. "Two Edge-Bonded Elastic Wedges of Different Materials and Wedge Angles under Surface Tractions," *Journal of Applied Mechanics*, pp. 377-386 (June 1971).

7. Hein, V. L. and F. Erdogan. "Stress Singularities in a Two-Material Wedge," *International Journal of Fracture*, 7(3) (Sept. 1971).

8. Westmann, R. A. "Geometrical Effects in Adhesive Joints," *Int. J. of Engrg. Sci.*, 13:369 (1975).

9. Luk, V. K. and L. M. Keer. "Stress Analysis for an Elastic Half Space Containing an Axially-Loaded Rigid Cylindrical Rod," *Int. J. of Solid and Structures*, 15:805-827 (1979).

10. Williams, M. L. "Stress Singularities Resulting from Various Boundary Conditions in Angular Corners of Plates in Extension," *J. Appl. Mech.*, *19, Trans. ASME*, *74*, p. 256 (1952).

11. Haritos, G. K. and L. M. Keer. "Stress Analysis for an Elastic Half Space Containing an Embedded Rigid Block," *Int. J. Solids and Structures*, 16:19-40 (1979).

12. Delale, F. "Stress Analysis of Multilayered Plates around Circular Holes," *Int. J. Engrg. Sci.*, 22(1):57-75 (1984).

13. Alblas, J. B. "Theorie Van de Driedimensionale Spanningstoestand in een Doorboorde Plaat," Ph.D. Thesis, H. J. Paris-Amsterdam (1957).

14. Folias, E. S. "The 3D Stress Field at the Intersection of a Hole and a Free Surface," *International Journal of Fracture* (1987).

15. Folias, E. S. "On the Three-Dimensional Theory of Cracked Plates," *Journal of Applied Mechanics*, 42(3):663-674 (1975).