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On the stress singularities at the intersection of a cylindrical inclusion with the free surface of a plate

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Abstract. Utilizing the form of a general 3D solution, the author investigates analytically the stress field in the neighborhood of the intersection of a cylindrical inclusion and a free surface. The inclusion is assumed to be of a homogeneous and isotropic material and is to be embedded in an isotropic plate of an arbitrary thickness. The stress field is induced by a uniform tension applied on the plate at points far remote from the inclusion (see Fig. 1).

The displacement and stress fields are derived explicitly and a stress singularity is shown to exist for the case when the inclusion is stiffer than the plate material. Moreover, the stress singularity is shown to be a function of the respective ratios of the shear moduli and Poisson's.

The special case of $G_2 \to 0$, $G_2 = G_1$ and $G_2 \to \infty$ are also investigated.

1. Introduction

Quite often in engineering practice, structures are composed of two elastic materials with different properties which are bonded together over some surface. Such type of problem has been investigated from a 2D point of view by many researchers and the results can be found in the literature. For example, in 1927 Knein [1] considered the plane strain problem of an orthogonal elastic wedge bonded to a rigid base. In 1955, Rongved [2] investigated the problem of two bonded elastic half-spaces subjected to a concentrated force in the interior. Subsequently, in 1959, Williams [3] studied the stress field around a fault or a crack in dissimilar media. The work was then generalized in 1965 by Rice and Sih [4] to include arbitrary angles.

It was not until 1968 that Bogy [5] considered the general problem of two bonded quarter-planes of dissimilar isotropic, elastic materials subjected to arbitrary boundary tractions. The problem was solved by an application of the Mellin transform in conjunction with the Airy stress function. In 1971, the same author [6] extended his work to also include dissimilar wedges of arbitrary angles. Shortly thereafter, Hein and Erdogan [7], using the same method of solution, independently reproduced the results by Bogy. Finally, in 1975 Westmann [8] studied the case of a wedge of an arbitrary angle which was bonded along a finite length to a half-space. His analysis showed the presence of two singularities close to each other. Thus, elimination of the first singular term does not lead to a bounded stress field since the second singularity is still present.

Based on 3D considerations,* in 1979 Luk and Keer [9] investigated the stress field in an elastic half space containing a partially embedded axially-loaded, rigid cylindrical rod. The

* Due to the symmetry of the applied load, the problem is mathematically 2D.

problem was formulated in terms of Hankel integral transforms and was finally cast into a system of coupled singular integral equations the solution of which was sought numerically. The authors were able, however, to extract in the limit from the integral equations the characteristic equation governing the singular behavior at the intersection of the free surface and that of the rigid inclusion. Their result was in agreement with that obtained by Williams [10] for a right-angle corner with fixed-free boundary conditions.

In 1980 Haritos and Keer [11] investigated the stress field in a half-space containing an embedded rigid block under conditions of plane strain. The problem was formulated by cleverly superimposing the solutions to the problem of horizontal and vertical line inclusions beneath an elastic half-space. By isolating the pertinent terms, the authors were able to extract directly from the integral equations the order of the stress singularity at both corners. Both results are in agreement with the Williams solution. Moreover, the authors point out the importance of the second singularity to the results of the load transfer problems.

Finally, in 1986 Folias [12], utilizing the form of a general 3D solution for the equilibrium of linear elastic layers which he developed in 1975 [13], derived explicitly the 3D displacement and stress fields at the intersection of a hole and a free surface. The analysis revealed that the stresses at the corner at proportional to $\varrho^{\alpha-2}$ where ϱ represents the local radius from the corner and $\alpha = 3.73959 \pm i1.11902$. It is interesting to note that the root is precisely the same as that obtained by Williams in his classic paper [10] for a 90 deg material corner with free–free stress boundaries (plane strain). An extension of the analysis to other angles of intersection revealed the same analogy between 3D and 2D. Thus the Williams solution has further applicability than was originally thought.

The same general 3D solution can now be used to solve the corresponding problem of a cylindrical inclusion. Moreover, the method is also applicable to non-symmetric applied loads while retaining a rather simple mathematical character.

2. Formulation of the problem

Consider the equilibrium of a homogeneous, isotropic, linear elastic plate which occupies the space $|x| < \infty$, $|y| < \infty$ and $|z| \leqslant h$ and contains a cylindrical inclusion of radius a whose generators are perpendicular to the bounding planes $z = \pm h$. It is assumed that the cylindrical inclusion is made of an isotropic and homogeneous material of different elastic properties than those of the plate. The plate is subjected to a uniform tensile load σ_0 in the direction of the y-axis and parallel to the bounding planes (see Fig. 1).

In the absence of body forces, the coupled differential equations governing the displacement functions $u_i^{(m)}$ are

$$\frac{1}{1-2\nu_m}\frac{\partial e^{(m)}}{\partial x_i} + \nabla^2 u_i^{(m)} = 0; \quad i = 1, 2, 3 \quad m = 1, 2, \tag{1}$$

where ∇^2 is the Laplacian operator, ν_m is Poisson's ratio, $u_i^{(1)}$ and $u_i^{(2)}$ represent the displacement functions in media 1 (plate) and 2 (inclusion) respectively, and

$$e^{(m)} \equiv \frac{\partial u_i^{(m)}}{\partial x_i} \quad i = 1, 2, 3; \quad m = 1, 2. \tag{2}$$

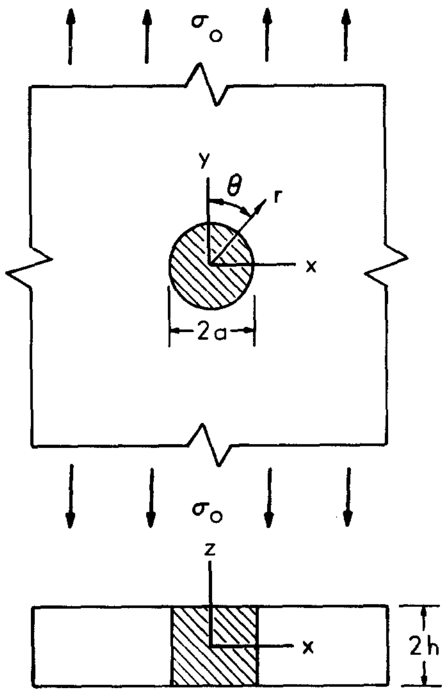

Fig. 1. Infinite plate of arbitrary thickness with cylindrical inclusion.

The stress-displacement relations are given by Hooke's law as

$$\sigma_{ij}^{(m)} = \lambda_m e_{kk}^{(m)} \delta_{ij} + 2Ge_{ij}^{(m)}, \tag{3}$$

where λ_m and G_m are the Lamé constants describing media 1 and 2.

3. Method of solution

The main objective of this analysis is to derive an asymptotic solution valid in the immediate vicinity of the corner points where the interface meets the free surface of the plate. For this purpose, we assume the complementary displacement field to be of the form [12, 13]:

$$u^{(m)} = \frac{1}{1-2v_m}\frac{\partial}{\partial x}\left\{2(1-v_m)f_2^{(m)} + h\frac{\partial f_1^{(m)}}{\partial z} + z\frac{\partial f_2^{(m)}}{\cdot \partial z}\right\} \tag{4}$$

$$v^{(m)} = \frac{1}{1-2v_m}\frac{\partial}{\partial y}\left\{2(1-v_m)f_2^{(m)} + h\frac{\partial f_1^{(m)}}{\partial z} + z\frac{\partial f_2^{(m)}}{\partial z}\right\} \tag{5}$$

$$w^{(m)} = \frac{1}{1-2v_m}\frac{\partial}{\partial z}\left\{-2(1-v_m)f_2^{(m)} + h\frac{\partial f_1^{(m)}}{\partial z} + z\frac{\partial f_2^{(m)}}{\partial z}\right\}, \tag{6}$$

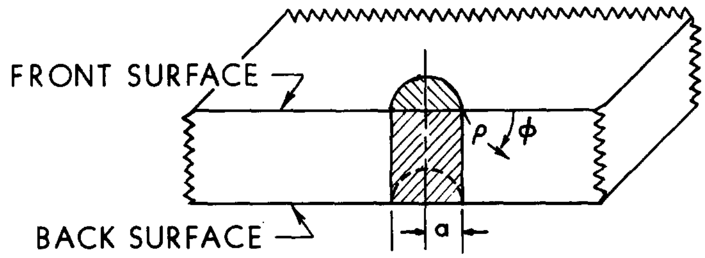

Fig. 2. Definition of local coordinates at the corner.

where the functions $f_1^{(m)}$ and $f_2^{(m)}$ are three dimensional harmonic functions. If we furthermore assume that

$$f_j^{(m)} = r^{-1/2} H_j^{(m)}(r - a, h - z) e^{i2\theta}; \quad j = 1, 2 \tag{7}$$

then the functions $H_j^{(m)}$ must satisfy the following equation:

$$\frac{\partial^2 H_j^{(m)}}{\partial (r - a)^2} + \frac{\partial^2 H_j^{(m)}}{\partial (h - z)^2} - \frac{15}{4(a + r - a)^2} H_j^{(m)} = 0. \tag{8}$$

It is found convenient at this stage to introduce the local coordinate system (see Fig. 2)

$$r - a = \varrho \cos \phi$$

$$h - z = \varrho \sin \phi$$

in view of which, (8) may now be written as

$$\frac{\partial^2 H_j^{(m)}}{\partial \varrho^2} + \frac{1}{\varrho} \frac{\partial H_j^{(m)}}{\partial \varrho} + \frac{1}{\varrho^2} \frac{\partial^2 H_j^{(m)}}{\partial \phi^2} - \frac{15 H_j^{(m)}}{4a^2[1 + (\varrho/a) \cos \phi]^2} = 0. \tag{9}$$

Under the assumption that the radius of the inclusion is sufficiently large, so that the condition $\varrho \ll a$ is meaningful, we seek the solution to (9) in the form

$$H_j^{(m)} = \sum_{n=0}^{\infty} \varrho^{\alpha+n} F_{jn}^{(m)}(\phi), \tag{10}$$

with α a constant. Without going into the mathematical details, we construct the following series expansion in ascending powers of ϱ:

$$\begin{aligned} H_j^{(m)} = {} & \varrho^{\alpha}\{A_j^{(m)} \cos(\alpha\phi) + B_j^{(m)} \sin(\alpha\phi)\} \\ & + \varrho^{\alpha+1}\{C_j^{(m)} \cos(\alpha + 1)\phi + D_j^{(m)} \sin(\alpha + 1)\phi\} + O(\varrho^{\alpha+2}), \end{aligned} \tag{11}$$

where the constants α, $A_j^{(m)}$, $B_j^{(m)}$, $C_j^{(m)}$ and $D_j^{(m)}$ are to be determined from the boundary conditions. Specifically,

$$\text{at } \phi = 0: \quad \sigma_{zz}^{(1)} = \tau_{xz}^{(1)} = \tau_{yz}^{(1)} = 0 \tag{12}$$

$$\text{at } \phi = \pi: \quad \sigma_{zz}^{(2)} = \tau_{xz}^{(2)} = \tau_{yz}^{(2)} = 0 \tag{13}$$

$$\text{at } \phi = \frac{\pi}{2}: \quad u_j^{(1)} = u_j^{(2)}; \quad j = 1, 2, 3 \tag{14}$$

$$\sigma_{rr}^{(1)} = \sigma_{rr}^{(2)}, \ \tau_{r\theta}^{(1)} = \tau_{r\theta}^{(2)}, \ \tau_{rz}^{(1)} = \tau_{rz}^{(2)}. \tag{15}$$

Substituting (11) into (12) and (13) one finds that all terms up to the order $O(\varrho^{\alpha-2})$ are satisfied if one assumes the following combinations vanish:

$$B_1^{(1)} = 0 \tag{16}$$

$$-h(\alpha + 1)A_1^{(1)} - B_2^{(1)} = 0 \tag{17}$$

$$A_1^{(2)} \sin(\alpha\pi) - B_1^{(2)} \cos(\alpha\pi) = 0 \tag{18}$$

$$[A_2^{(2)} - h(\alpha + 1)B_1^{(2)}] \tan(\alpha\pi) - B_2^{(2)} + h(\alpha + 1)A_1^{(2)} = 0. \tag{19}$$

Similarly, the displacement and stress boundary conditions at the cylindrical surface are satisfied if we assume the following combinations vanish

$$\begin{aligned} &\frac{1}{1 - 2\nu_1}\left\{(-\alpha - 1 + 2\nu_1)A_2^{(1)} \tan\left(\frac{\alpha\pi}{2}\right) - (-\alpha - 1 + 2\nu_1)B_2^{(1)} \right. \\ &\qquad \left. + h(\alpha + 1)\left[A_1^{(1)} + B_1^{(1)} \tan\left(\frac{\alpha\pi}{2}\right)\right]\right\} \\ &- \frac{1}{1 - 2\nu_2}\left\{(-\alpha - 1 + 2\nu_2)\left[A_2^{(2)} \tan\left(\frac{\alpha\pi}{2}\right) - B_2^{(2)}\right] \right. \\ &\qquad \left. + h(\alpha + 1)\left[A_1^{(2)} + B_1^{(2)} \tan\left(\frac{\alpha\pi}{2}\right)\right]\right\} = 0 \end{aligned} \tag{20}$$

$$\begin{aligned} &\frac{1}{1 - 2\nu_1}\left\{(\alpha - 2 + 2\nu_1)A_2^{(1)} + (\alpha - 2 + 2\nu_1)B_2^{(1)} \tan\left(\frac{\alpha\pi}{2}\right) \right. \\ &\qquad \left. + h(\alpha + 1)\left[A_1^{(1)} \tan\left(\frac{\alpha\pi}{2}\right) - B_1^{(1)}\right]\right\} \end{aligned}$$

$$- \frac{1}{1 - 2\nu_2}\left\{(\alpha - 2 + 2\nu_2)\left[A_2^{(2)} + B_2^{(2)} \tan\left(\frac{\alpha\pi}{2}\right)\right]\right.$$

$$\left. + h(\alpha + 1)\left[A_1^{(2)} \tan\left(\frac{\alpha\pi}{2}\right) - B_1^{(2)}\right]\right\} = 0 \tag{21}$$

$$\alpha A_2^{(1)} - (1 - \alpha) \tan\left(\frac{\alpha\pi}{2}\right) B_2^{(1)} - \beta\left\{\alpha A_2^{(2)} + \alpha \tan\left(\frac{\alpha\pi}{2}\right) B_2^{(2)}\right.$$

$$\left. - \tan\left(\frac{\alpha\pi}{2}\right)\left[A_2^{(2)} \sin(\alpha\pi) - B_2^{(2)} \cos(\alpha\pi)\right]\right\} = 0 \tag{22}$$

$$-(\alpha - 1)A_2^{(1)} \tan\left(\frac{\alpha\pi}{2}\right) + (\alpha - 2)B_2^{(1)} - \beta\left\{\left[-(\alpha - 1) \tan\left(\frac{\alpha\pi}{2}\right)\right.\right.$$

$$\left.\left. + \sin(\alpha\pi)\right] A_2^{(2)} + [(\alpha - 1) - \cos(\alpha\pi)]B_2^{(2)}\right\} = 0, \tag{23}$$

where for simplicity we have defined

$$\beta \equiv \frac{1 - 2\nu_1}{1 - 2\nu_2}\frac{G_2}{G_1}. \tag{24}$$

The characteristic value α may now be determined by setting the determinant of the algebraic system (20)–(24) equal to zero. Once the roots have been determined, the complete displacement and stress fields can be constructed in ascending powers of ϱ.

4. Discussion of the results

Without going into the mathematical details, the characteristic values α can easily be determined with the aid of a computer. Although the equation has an infinite number of complex roots, only the one with a $1 < \min \operatorname{Re} \alpha < 2$ is relevant. In general, the characteristic values of α depend on the material properties of both the plate as well as the inclusion.

The analysis clearly shows that, in the neighborhood of the interface and the free surface, the stress field is proportional to $\varrho^{\alpha-2}$ and that for certain material properties it is singular. Moreover, the first root is found to be precisely the same as that of the corresponding 2D case [6]. Figures 3, 4 and 5 depict typical results for various material properties. Finally, in the limit as the shear modulus (i) $G_2 \to 0$ and (ii) $G_2 \to \infty$ one recovers the results corresponding to a (i) hole (i.e., $\alpha = 3.73959 \pm i1.11902$) and (ii) a perfectly rigid inclusion (i.e., $\alpha = 1.7112 + i0.$), respectively. It is interesting to note that in both limit cases the exponent α is the same as that obtained by Williams [10] for a 90 degree material angle with (i) free–free and (ii) fixed-free boundaries in plane strain. Finally, as $G_2 \to G_1$ and $\nu_2 \to \nu_1$, the solution of a continuous plate is recovered, a result that clearly meets our expectations.

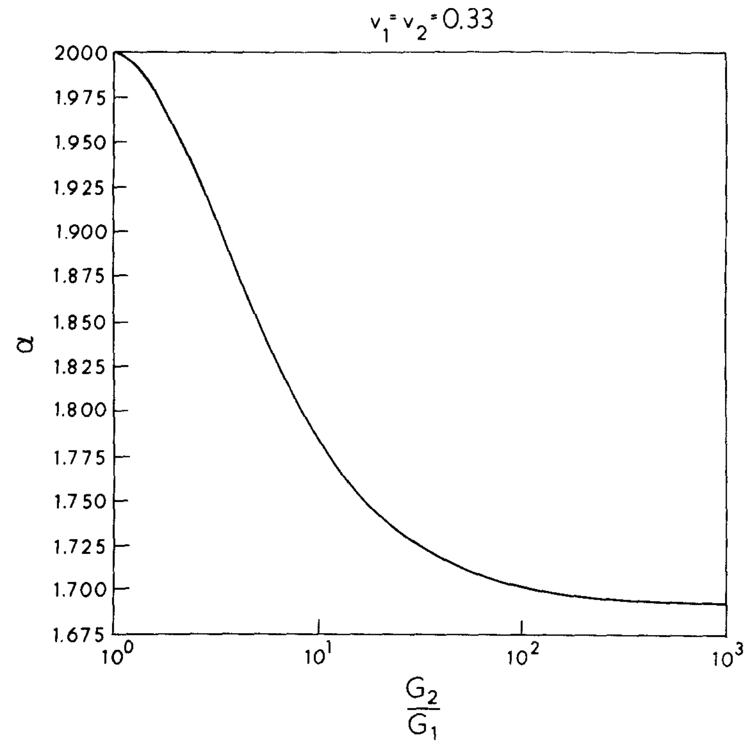

Fig. 3. Strength of the singularity vs $G_2 \backslash G_1$ for $\nu_1 = \nu_2 = 0.33$.

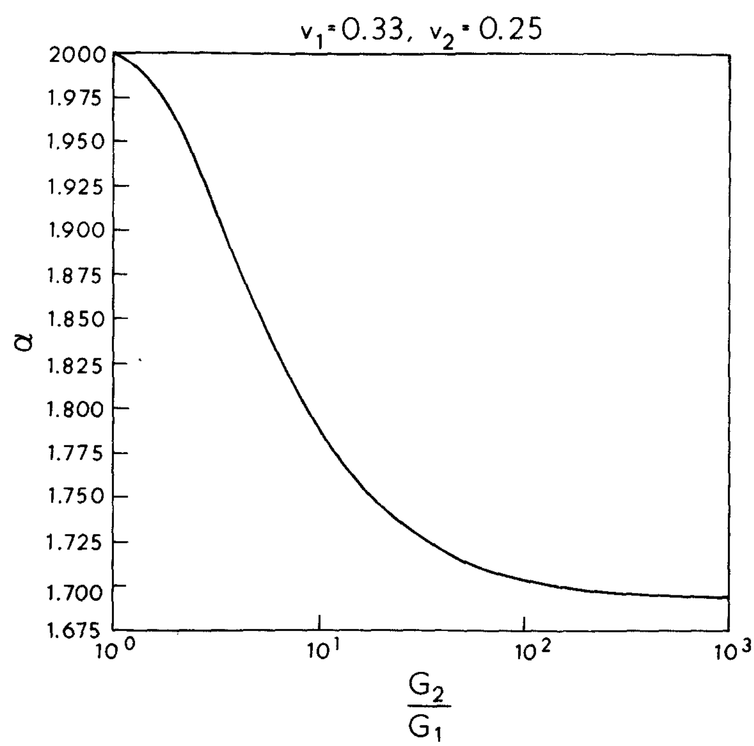

Fig. 4. Strength of the singularity vs $G_2 \backslash G_1$ for $\nu_1 = 0.33$, $\nu_2 = 0.25$.

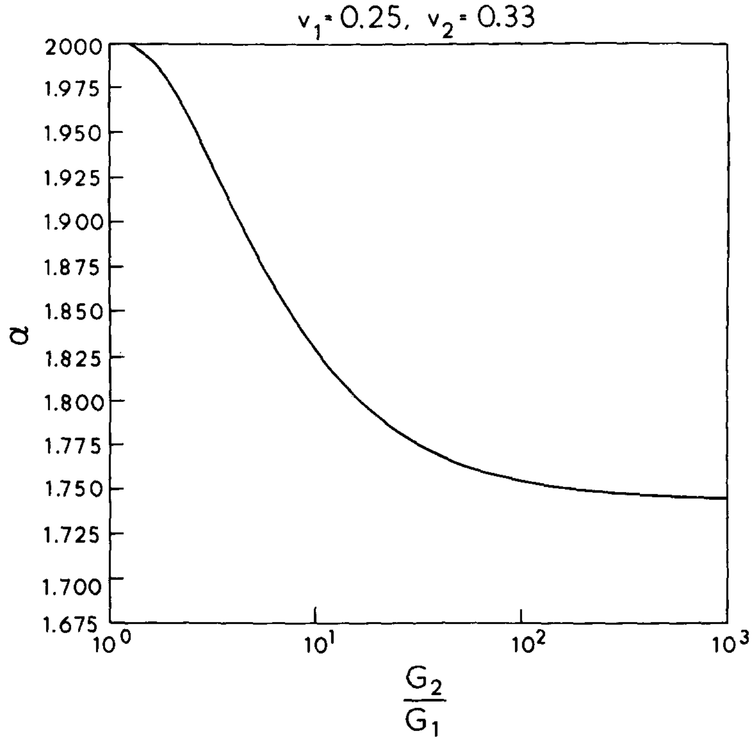

Fig. 5. Strength of the singularity vs $G_2 \backslash G_1$ for $\nu_1 = 0.25$, $\nu_2 = 0.33$.

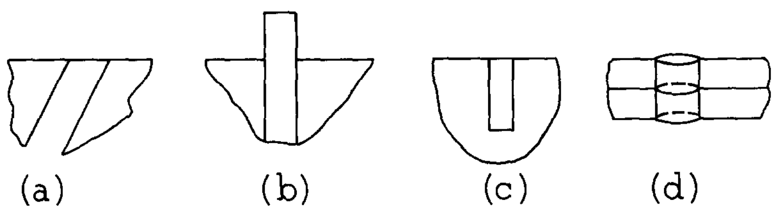

Fig. 6. Geometrical applications.

An extension of this analysis to other angles of intersection with the free surface reveals the same results as those predicted by Bogy [6] for the case of plane strain. A few cases of practical interest which come to mind are shown in Fig. 6(a)–(c). Similarly, it can be shown that the same results apply at the intersection of an interface and the free surface of a hole in a laminated plate consisting of homogeneous and isotropic laminates (see Fig. 6d). Finally, by way of a conjecture, one may now deduce that the same analogy exists for the corresponding cases consisting of anisotropic materials.

It should also be noted that the analysis confirms the presence of another (a little bit weaker) singularity which was first pointed out by Westmann [8]. While it is true that the singular stress field is dominated by the largest singularity, the presence of two singular terms has important implications to the problem of adhesion as well as the problem of load diffusion from one material to another [11].

The results of this analysis are also of importance to the field of composite materials. In general, composite structures are designed in such a way as to carry the load along the

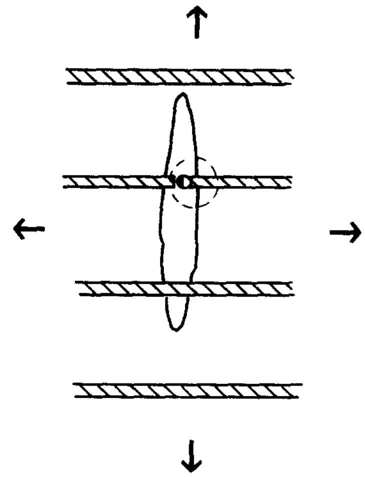

Fig. 7. Possible composite failure mode.

direction of their fibers. Quite often, however, a small portion of that load will be applied, at least locally, in a direction perpendicular to the fibers (e.g., a pressurized vessel). As a practical matter, let us consider a region where the matrix and a few fibers have cracked in a manner depicted in Fig. 7. Conditions in the neighborhood of the circled fiber are very similar* to those assumed in the present paper. There are, however, two important differences. First, the dense distribution of fibers induces stress interactions which lead to higher stress levels. Second, the typical diameter of a fiber is approximately 0.003 inches, a magnitude that may be thought of as of the same order as ϱ. In that case, (9) is no longer separable and a different form of the solution must be sought.**

In closing, it is noteworthy to point out the importance of the general 3D solution [13] for it reveals the inherent form of the solution at such neighborhoods and permits a simple analytical approach, à la Williams [10], for the determination of the stress singularities.

Acknowledgement

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* Here the author assumed that the same analogy between 3D and 2D exists for anisotropic materials, too.
** This matter is presently under investigation.

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Résumé. En utilisant la forme d'une solution générale à trois dimensions, l'auteur analyse le champ de contraintes au voisinage de l'intersection d'une inclusion cylindrique et d'une surface libre. On suppose que l'inclusion est constituée d'une matériau homogène et isotrope, et qu'elle est insérée dans une plaque isotrope d'épaisseur arbitraire. Le champ de contraintes et dû à une tension uniforme appliquée en des points suffisamment éloignés de l'inclusion.

On exprime de manière explicite les champs de déplacements et de contraintes, et on montre qu'il existe une singularité de la contrainte dans le cas où l'inclusion est dans un matériau plus rigide que celui de la plaque. En outre, on montre que la singularité de la contrainte est fonction des rapport respectifs des modules de cisaillement et de Poisson.

On étudie également les cas spéciaux $G_2 \to 0$, $G_2 = G_1$ et $G_1 \to \infty$.