International Journal of Fracture 39: 121-127 (1989)

Viscoplastic flow due to penetration: a free boundary value problem

DANG DINH ANG, TIM FOLIAS, FRITZ KEINERT and FRANK STENGER
Department of Mathematics, University of Utah, Salt Lake City, Utah 84112, USA

Received 10 December 1987; accepted in revised form 1 April 1988

Abstract. Under the action of a pressure gradient, a solid body B penetrates into another body. Body B is assumed to be of an incompressible, viscoplastic, Bingham material. As a first model, the problem may be treated one-dimensionally in the space variable x as well as the time variable t.

By utilizing the Green's function, the location of the moving boundary $s(t)$, i.e., the boundary between the region of viscoplastic flow and the core, is expressed in terms of an integral equation, the solution of which may then be sought numerically.

1. Formulation of the problem

A solid body B of width $2H$ and under the action of a pressure gradient, penetrates into another body, in an action similar to that of a bullet entering an object. We assume that body B is an incompressible viscoplastic Bingham body, that is, it satisfies Bingham's law

$$\tau - \tau_0 = \pm\mu \frac{\partial u}{\partial x}, \tag{1}$$

where τ_0 is the yield stress, μ the coefficient of viscosity and u the velocity in the y-direction. The movement is in the y-direction only and is assumed to be independent of z and symmetric about the plane $x = H$ (see Fig. 1).

The body B is divided into two parts

$$B_1 = \{x\colon |x| < s(t) \quad \text{or} \quad |x| > 2H - s(t)\}$$

$$B_2 = \{x\colon s(t) \leqslant |x| \leqslant 2H - s(t)\}.$$

In B_1 (resp. B_2) the tangential stress is larger (resp. smaller) than the yield stress τ_0. We call B_1 the zone of viscoplastic flow and B_2 the core.

In the zone of viscoplastic flow, the velocity $u(x, t)$ satisfies the diffusion equation* (see Rubinstein [2], Chapter 4)

$$k^2 \frac{\partial u}{\partial t} + \frac{1}{\nu\varrho}\frac{\partial p}{\partial y} = \frac{\partial^2 u}{\partial x^2}, \quad k^2 = \nu^{-1}, \tag{2}$$

* The reader should notice that at the interface $x = s(t)$, $\partial u/\partial x = 0$.

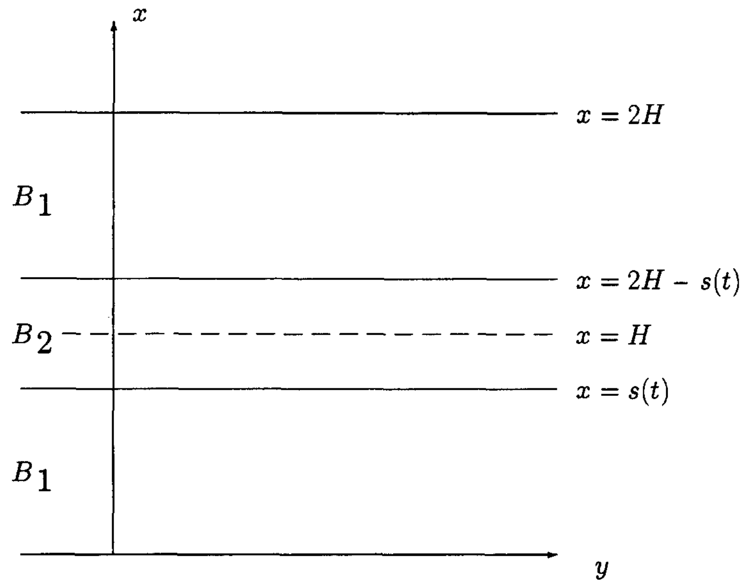

Fig. 1. Geometrical configuration.

where $\partial p/\partial y$ is the pressure gradient in the y-direction and may be interpreted as a driving force, ϱ the (constant) density and ν the kinematic viscosity. Due to symmetry, it is sufficient to consider (2) in the domain $0 < x < s(t)$, and furthermore we assume that

$$u(0, t) = f(t), \tag{3}$$

$$\frac{1}{\varrho}\frac{\partial p}{\partial y} = g(t), \tag{4}$$

where $f(t)$ and $g(t)$ are given functions. Since the core is rigid, the velocity in it is

$$u = u_0(t) = u(s(t), t), \tag{5}$$

where it is assumed that $u_0(0) \neq 0$. At the interface $x = s(t)$, the tangential stress is equal to the yield stress, and hence by (1) we must have

$$\frac{\partial u}{\partial x} = 0 \quad \text{at} \quad x = s(t). \tag{6}$$

The problem therefore is to determine $s(t)$ from the above conditions as well as the following

$$s(0) = b > 0 \tag{7}$$

$$u(x, 0) = \phi(x), \quad \phi(0) = f(0) \tag{8}$$

$$\dot{u}_0(t) = -g(t) - \frac{\tau_0}{\varrho(H - s(t))}, \tag{9}$$

where the dot on top of a function indicates differentiation and (9) follows from a consideration of the forces acting on the core (e.g., see [2]).

Since by (6)

$$\dot{u}_0(t) = \frac{\mathrm{d}}{\mathrm{d}t}[u(s(t), t)] = u_x(s(t), t)\dot{s}(t) + u_t(s(t), t) = u_t(s(t), t),$$

we obtain from (2), (4), by letting $x \to s(t)$, that

$$\dot{u}_0(t) = -g(t) + \nu u_{xx}(s(t), t), \tag{10}$$

which upon comparing it with (9) gives

$$u_{xx}(s(t), t) = -\frac{\tau_0}{\nu\varrho(H - s(t))}. \tag{11}$$

Moreover, in order to be compatible with our previous assumptions, at $t = 0$ we must require that

$$\ddot{\phi}(b) = -\frac{\tau_0}{\nu\varrho(H - b)}. \tag{12}$$

Notice that in this analysis we have for simplicity assumed that $s(0) > 0$. The case $s(0) = 0$ requires some special mathematical rigor which for the sake of brevity we will omit. Perhaps it is appropriate at this point to comment on the difference between the present problem and the classical Stefan problem. For the classical Stefan problem, the location of the moving boundary, $x = s(t)$, is governed by the velocity u as well as its derivative with respect to x, whereas in the present problem it is also governed by the time derivative, i.e., an additional constraint which makes the solution even more difficult.

For the solution of the problem, we shall use the method of Green's functions. However, before engaging in the details of the construction of the solution, we first define the concept of a solution to our problem. By a solution to our problem, henceforth called the FBP, is meant an ordered pair $u(x, t)$, $s(t)$ of functions, $u(x, t)$ defined on $0 \leqslant x \leqslant s(t)$, $0 \leqslant t \leqslant \sigma$, $s(t)$ defined on $0 \leqslant t \leqslant \sigma$, for some $\sigma > 0$, such that

(i) u_{xx}, u_t are continuous in $0 \leqslant x \leqslant s(t)$ for $0 < t < \sigma$
(ii) u and u_x are continuous for $0 \leqslant x \leqslant s(t)$, $0 \leqslant t \leqslant \sigma$
(iii) u satisfies (2) in $0 < x < s(t)$, $0 < t \leqslant \sigma$
(iv) conditions (3)–(9) are satisfied
(v) is Lipschitzian on $(0, \sigma]$.

2. Method of solution

We shall formulate the problem in terms of an integral equation, and for this purpose we shall require some regularity conditions on the initial and boundary data:

(vi) $f(t)$ is continuous, $g(t)$ is C^1 on $t \geqslant 0$
(vii) $\phi(x)$ is C^2 on $(0, b)$, and the left-hand derivative $\dot{\phi}(b)$ exists
(viii) $s(t)$ is C^1 for $t \geqslant 0$.

We now define

$$K(x, t; \xi, \tau) = \frac{k}{2\sqrt{\pi}} \frac{1}{\sqrt{t - \tau}} \exp\left(-\frac{k^2(x - \xi)^2}{4(t - \tau)}\right),$$

and

$$G(x, t; \xi, \tau) = K(x, t; \xi, \tau) - K(x, t; -\xi, \tau)$$

$$N(x, t; \xi, \tau) = K(x, t; \xi, \tau) + K(x, t; -\xi, \tau)$$

$$0 < x < s(t), \quad 0 < \xi < s(\tau), \quad 0 < \tau < t.$$

These are the Green's functions we shall use. For their various properties the reader is referred to Friedman [1] (chapter on free boundary value problems) or to Rubinstein [2]. We shall use them freely in our subsequent analysis without explicit mention of the references.

Thus, let $u(x, t)$, $s(t)$ be a solution of our FBP. Integrating the identity

$$(Gu_\xi - G_\xi u)_\xi - k^2(Gu)_\tau = k^2 Gg \tag{13}$$

over the region $\{(\xi, \tau): 0 \leqslant \xi \leqslant s(\tau), \varepsilon \leqslant \tau \leqslant t - \varepsilon\}$, applying Green's identity and letting $\varepsilon \to 0$, we obtain

$$\begin{aligned} u(x, t) = {} & \int_0^b \phi(\xi) G(x, t; \xi, 0) d\xi - \frac{1}{k^2} \int_0^t u_0(\tau) G_\xi(x, t; s(\tau), \tau) d\tau \\ & + \int_0^t u_0(\tau) G(x, t; s(\tau), \tau) \dot{s}(\tau) d\tau + \frac{1}{k^2} \int_0^t f(\tau) G_\xi(x, t; 0, \tau)\, d\tau \\ & - \int_0^t \int_0^{s(\tau)} G(x, t; \xi, \tau) d\xi g(\tau) d\tau \quad 0 < x < s(t), \quad t > 0. \end{aligned} \tag{14}$$

We now differentiate both sides of (14) with respect to x for $0 < x < s(t)$

$$\begin{aligned} u_x(x, t) = {} & \int_0^b \phi(\xi) G_x(x, t; \xi, 0) d\xi - \int_0^t u_0(\tau) N_\tau(x, t; s(\tau), \tau) d\tau \\ & + \int_0^t u_0(\tau) G_x(x, t; s(\tau), \tau) \dot{s}(\tau) d\tau + \int_0^t f(\tau) N_\tau(x, t; 0, \tau) d\tau \\ & - \int_0^t \int_0^{s(\tau)} G_x(x, t; \xi, \tau) d\xi g(\tau) d\tau, \quad 0 < x < s(t), \quad t > 0, \end{aligned} \tag{15}$$

where we have made use of the identity

$$G_{x\xi} = k^2 N_\tau .$$

Integrating by parts, we have

$$\int_0^b \phi(\xi)G_x(x, t; \xi, 0)\mathrm{d}\xi = -\int_0^b \phi(\xi)N_\xi(x, t; \xi, 0)\mathrm{d}\xi$$

$$= \phi(0)N(x, t; 0, 0) - \phi(b)N(x, t; b, 0)$$

$$+ \int_0^b \dot{\phi}(\xi)N(x, t; \xi, 0)\mathrm{d}\xi \tag{16}$$

$$-\int_0^t u_0(\tau)N_\tau(x, t; s(\tau), \tau)\mathrm{d}\tau$$

$$= \int_0^t u_0(\tau)\left[N_\xi(x, t; s(\tau), \tau)\dot{s}(\tau) - \frac{\mathrm{d}}{\mathrm{d}\tau}N(x, t; s(\tau), \tau)\right]\mathrm{d}\tau$$

$$= -\int_0^t u_0(\tau)G_x(x, t; s(\tau), \tau)\dot{s}(\tau)\mathrm{d}\tau - u_0(t)N(x, t, s(t), t)$$

$$+ u_0(0)N(x, t; b, 0) + \int_0^t \dot{u}_0(\tau)N(x, t; s(\tau), \tau)\mathrm{d}\tau \tag{17}$$

$$\int_0^t f(\tau)N_\tau(x, t; 0, \tau)\mathrm{d}\tau = -f(0)N(x, t; 0, 0) - \int_0^t \dot{f}(\tau)N(x, t; 0, \tau)\mathrm{d}\tau \tag{18}$$

$$\int_0^{s(\tau)} G_x(x, t; \xi, \tau)\mathrm{d}\xi = -\int_0^{s(\tau)} N_\xi(x, t; \xi, \tau)\mathrm{d}\xi$$

$$= -N(x, t; s(\tau), \tau) + N(x, t; 0, \tau). \tag{19}$$

Substituting (16)–(19) and (9) into (15) and simplifying yields

$$u_x(x, t) = \int_0^b \dot{\phi}(\xi)N(x, t; \xi, 0)\mathrm{d}\xi - \frac{\tau_0}{\varrho}\int_0^t \frac{1}{H - s(\tau)} N(x, t; s(\tau), \tau)\mathrm{d}\tau$$

$$-\int_0^t [\dot{f}(\tau) + g(\tau)]N(x, t; 0, \tau)\mathrm{d}\tau \tag{20}$$

and upon letting $x \nearrow s(t)$, by (6)

$$0 = \int_0^b \dot{\phi}(\xi)N(s(t), t; \xi, 0)\mathrm{d}\xi - \frac{\tau_0}{\varrho}\int_0^t \frac{1}{H - s(\tau)} N(s(t), t; s(\tau), \tau)\mathrm{d}\tau$$

$$-\int_0^t [\dot{f}(\tau) + g(\tau)]N(s(t), t; 0, \tau)\mathrm{d}\tau. \tag{21}$$

Consider what happens in (21) as $t \to 0$. Splitting up the first integral by using $N(s(t), t; \xi, 0) = K(s(t), t; \xi, 0) + K(s(t), t; -\xi, 0)$ and likewise the second one, we observe that three of the resulting five integrals tend to zero exponentially. The other two have leading terms of $\sqrt{t}$, which of course have to cancel. Considering only the leading terms, we find

$$s(t) - s(\tau) \approx \dot{s}(t)(t - \tau)$$

$$K(s(t), t; s(\tau), \tau) \approx \frac{k}{2\sqrt{\pi}}(t - \tau)^{-1/2}$$

so

$$\frac{\tau_0}{\varrho}\int_0^t \frac{1}{H - s(\tau)} K(s(t), t; s(\tau), \tau)\mathrm{d}\tau \approx \frac{\tau_0}{\varrho}\frac{k}{\sqrt{\pi}}\frac{1}{H - s(t)} t^{1/2}. \tag{22}$$

Also,

$$\begin{aligned}\int_0^b K(s(t), t; \xi, 0)\mathrm{d}\xi &= \frac{1}{\sqrt{\pi}}\int_{k(s(t)-b)/2\sqrt{t}}^{ks(t)/2\sqrt{t}} \mathrm{e}^{-x^2}\mathrm{d}x \\ &\approx 1/2 - \frac{k\dot{s}(0)}{2\sqrt{\pi}} t^{1/2}\end{aligned}$$

and

$$\begin{aligned}\int_0^b (\xi - s(t))K(s(t), t; \xi, 0)\mathrm{d}\xi &\approx \frac{\sqrt{t}}{k\sqrt{\pi}}[\mathrm{e}^{-k^2 s(t)^2/4t} - \mathrm{e}^{-k^2\dot{s}(0)^2 t}] \\ &\approx -\frac{1}{k\sqrt{\pi}} t^{1/2}.\end{aligned}$$

As $t \to 0$, $K(s(t), t; \xi, 0)$ behaves like an approximate δ-function with peak near b. Thus, if h is a C^1-function, then

$$\begin{aligned}\int_0^b h(\xi)K(s(t), t; \xi, 0)\mathrm{d}\xi &\approx \int_0^b [h(b) + \dot{h}(b)(\xi - s(t) + \dot{s}(t)t)]K(s(t), t; \xi, 0)\mathrm{d}\xi \\ &\approx \tfrac{1}{2}h(b) - \frac{1}{\sqrt{\pi}k}\left[\frac{k^2\dot{s}(0)}{2}h(b) - \dot{h}(b)\right] t^{1/2}.\end{aligned} \tag{23}$$

In particular, if $h = \phi$, we find

$$\int_0^b \phi(\xi)K(s(t), t; \xi, 0)\mathrm{d}\xi \approx \frac{\dot{\phi}(b)}{k\sqrt{\pi}} t^{1/2}. \tag{24}$$

Thus, as mentioned above, the consistency of (21) as t approaches zero requires condition (12). The rough estimates above can be made more rigorous and lead to the conclusion that if the given functions f, g, ϕ are smooth and (12) is violated, s cannot be Lipschitzian at $t = 0$. A sharp corner in the moving boundary is expected in this case.

3. Numerical method

Assuming that (12) is satisfied and $s(t)$ is smooth, we find

$$N(s(t), t; s(\tau), \tau) = \frac{k}{2\sqrt{\pi}}(t - \tau)^{-1/2} + F(s(t), t; s(\tau), \tau),$$

where F is smooth for $\tau < t$, and

$$F(s(t), t; s(\tau), \tau) = O((t - \tau)^{1/2}) \quad \text{as} \quad \tau \to t.$$

Thus we have

$$\frac{k}{2\sqrt{\pi}}\frac{\tau_0}{\varrho}\int_0^t \frac{1}{H - s(\tau)}(t - \tau)^{-1/2}\mathrm{d}\tau = \int_0^b \phi(\xi)N(s(t), t; \xi, 0)\mathrm{d}\xi$$

$$- \frac{\tau_0}{\varrho}\int_0^t \frac{1}{H - s(\tau)} F(s(t), t; s(\tau), \tau)\mathrm{d}\tau$$

$$- \int_0^t [\dot{f}(\tau) + g(\tau)]N(s(t), t; 0, \tau)\mathrm{d}\tau, \tag{25}$$

The integral equation may now be solved by an iteration scheme. Starting with an initial guess $s^{(0)}(t)$ for the moving boundary, for example $s^{(0)}(t) \equiv b$, we can substitute the ith iterate $s^{(i)}(t)$ into the right-hand side of (25) and calculate a new approximation $s^{(i+1)}(t)$.

We are now in the process of testing this method and plan to publish the results in a subsequent paper.

Acknowledgements

Part of this research was supported by U.S. Army research contract No. DAAL03-87-K-0008. The authors gratefully acknowledge this support.

References

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2. L.I. Rubinstein, *The Stefan Problem*, Translations of Mathematical Monographs, 27, American Mathematical Society, Providence (1970).

Résumé. On traite le cas d'un solide B, supposé incompressible, viscoplastique et en matériau de Bingham, dans un autre corps sous l'effet d'un gradient de pression. En première analyse, le problème peut être traité suivant une dimension, sur une variable d'espace x ou de temps t.

En recourant à une fonction de Green, on exprime sous forme d'une équation intégrale la position de la frontière en mouvement $s(t)$, à savoir la frontière entre la région d'écoulement viscoplastique et la portion dure. La solution de cette équation peut être trouvée par voie numérique.