International Journal of Fracture 35: 187–194 (1987)

The 3D stress field at the intersection of a hole and a free surface

E.S. FOLIAS
Department of Civil Engineering, University of Utah, Salt Lake City, UT 84112, USA

Received 21 January 1987; accepted in revised form 1 April 1987

Abstract. The author in this paper investigates the analytical stress field in the neighborhood of the intersection of a hole and a free boundary. Utilizing the form of a general 3D solution, which was constructed in a previous paper, he recovers the explicit displacement and stress fields in this neighborhood. Moreover, the analysis shows the complementary solution to be proportional to $\varrho^{\alpha-2}$, where α is independent of the Poisson's ratio ν. Interestingly enough the first root is exactly that obtained by the William's solution for a 90 deg corner with free–free of stress boundaries.

1. Introduction

It is well known that at the vertex of a sector plate, in stretching and in bending, unbounded stresses may occur for certain vertex angles. In the case of 2D problems, this has been investigated analytically by Williams [1,2] subject to various edge conditions. In 3D, however, far less is known about such problems particularly when the re-entrant angle is other than 2π. To the best of the author's knowledge, an explicit analytical solution for the determination of the stress singularity in the neighbourhood of the intersection between the crack front and the free surface has yet to be constructed. Various past analyses suggest four different types of stress singularities at such neighborhoods. A somewhat complete historical discussion can be found in [3].

In this paper the author investigates the solution of a simpler problem, that of a thick plate which has been weakened by the presence of a circular hole. While it is true that such an analysis will not directly provide us with the singularity strength of the 3D Griffith crack problem, it will evince many characteristics of the solution and provide guidance for the ultimate construction of such a solution. Thus the circular hole problem is a logical fountainhead for detailed theoretical study.

2. Formulation of the problem

Consider the equilibrium of a homogeneous, isotropic, elastic plate that occupies the space $|x| < \infty$, $|z| \leqslant h$ and contains a cylindrical hole of a radius a whose generators are perpendicular to the bounding planes, namely $z = \pm h$. Let the plate be subjected to a uniform tensile load σ_0 along the y-axis and parallel to the bounding planes (see Fig. 1).

In the absence of body forces, the coupled differential equations governing the displacement functions u, v and w are

$$\frac{m}{m-2}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) e + \nabla^2(u, v, w) = 0 \qquad (1)–(3)$$

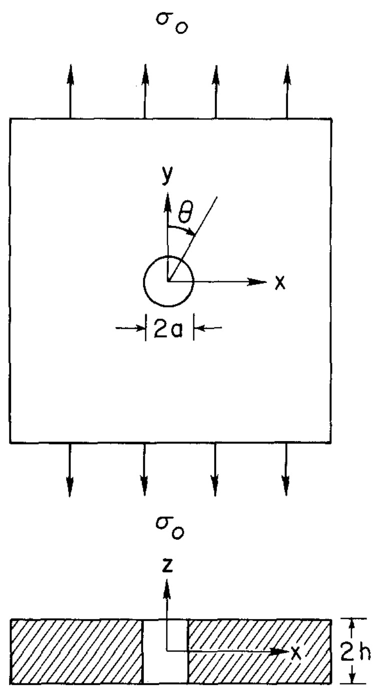

Fig. 1. Geometrical configuration of a plate weakened by a circular hole of radius a.

where ∇^2 represents the Laplacian operator, $m \equiv 1/\nu$, ν is Poisson's ratio,

$$e \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \tag{4}$$

and the stress-displacement relations are given by Hooke's law as:

$$\sigma_{xx} = 2G\left\{\frac{\partial u}{\partial x} + \frac{e}{m-2}\right\}, \ldots, \tau_{xy} = G\left\{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right\}, \ldots \tag{5)–(10}$$

with G being the shear modulus.

3. Method of solution

The main objective of this analysis is to construct an asymptotic solution which is valid in the immediate vicinity of the corner point, i.e., the neighborhood where the hole surface intersects the plate surface $z = +h$. Guided by a general solution to the equilibrium of elastic layers which the author constructed in a previous paper [4], we assume the complementary displacement field in the form:

$$u = \frac{1}{m-2}\frac{\partial}{\partial x}\left\{2(m-1)f_2 + mh\frac{\partial f_1}{\partial z} + mz\frac{\partial f_2}{\partial z}\right\} + \frac{\partial g}{\partial y} \tag{11}$$

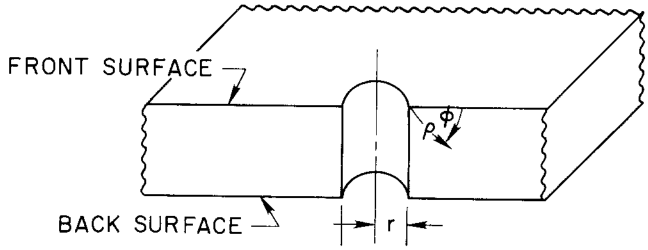

Fig. 2. Definition of local coordinates at the corner.

$$v = \frac{1}{m-2}\frac{\partial}{\partial y}\left\{2(m-1)f_2 + mh\frac{\partial f_1}{\partial z} + mz\frac{\partial f_2}{\partial z}\right\} - \frac{\partial g}{\partial x} \tag{12}$$

$$w = \frac{1}{m-2}\frac{\partial}{\partial z}\left\{-2(m-1)f_2 + mh\frac{\partial f_1}{\partial z} + mz\frac{\partial f_2}{\partial z}\right\} \tag{13}$$

where the functions f_1, f_2 and g represent three-dimensional harmonic functions. Utilizing the local cylindrical coordinates, we furthermore assume the harmonic functions to be of the form

$$f_i = r^{-1/2}H_i(r-a, h-z)\cos(2\theta); \quad i = 1, 2 \tag{14}$$

and

$$g = r^{-1/2}H_3(r-a, h-z)\sin(2\theta), \tag{15}$$

where the newly defined functions H_i now satisfy the differential equation

$$\frac{\partial^2 H_i}{\partial(r-a)^2} + \frac{\partial^2 H_i}{\partial(h-z)^2} - \frac{15}{4(a+r-a)^2}H_i = 0; \quad i = 1, 2, 3. \tag{16}$$

Moreover, it is found convenient at this stage to introduce the local coordinate system to the corner point (see Fig. 2), i.e.,

$$r - a = \varrho\cos\phi \tag{17}$$

$$h - z = \varrho\sin\phi, \tag{18}$$

which transforms equation (16) into

$$\frac{\partial^2 H_i}{\partial\varrho^2} + \frac{1}{\varrho}\frac{\partial H_i}{\partial\varrho} + \frac{1}{\varrho^2}\frac{\partial^2 H_i}{\partial\phi^2} - \frac{15}{4a^2[1 + (\varrho/a)\cos\phi]^2}H_i = 0. \tag{19}$$

Assuming, therefore, that the radius of the hole a is sufficiently large so that $\varrho \ll a$, one may now seek the solution to (19) in the form

$$H_i = \sum_{n=0}^{\infty}\varrho^{\alpha+n}F_n(\phi) \tag{20}$$

where α represents a constant. Without going into the mathematical details, the usual asymptotic analysis reveals that H_i must be of the form

$$H_i = \varrho^{\alpha}\{A_0^{(i)} \cos(\alpha\phi) + B_0^{(i)} \sin(\alpha\phi)\} + \varrho^{\alpha+1}\{A_1^{(i)} \cos(\alpha + 1)\phi + B_1^{(i)} \sin(\alpha + 1)\phi\}$$
$$+ 0(\varrho^{\alpha+3}): \quad i = 1, 2, 3, \tag{21}$$

where the constants α, $A_0^{(i)}$, $A_1^{(i)}$, $B_0^{(i)}$, $B_1^{(i)}$ etc. are to be determined from the boundary conditions which are that the stresses along the surfaces $\phi = 0$ and $\phi = \pi/2$ must vanish for all ϱ.

Specifically, the boundary conditions on the surface $z = h$ can be shown to be satisfied if the following combinations vanish

$$\frac{m}{m-2}\left\{h\frac{\partial^3 f_1}{\partial z^3} + z\frac{\partial^3 f_2}{\partial z^3}\right\} = 0 \tag{22}$$

$$\frac{2m}{m-2}\frac{\partial}{\partial r}\left\{\frac{\partial f_2}{\partial z} + h\frac{\partial^2 f_1}{\partial z^2} + z\frac{\partial^2 f_2}{\partial z^2}\right\} - \frac{2}{r}\frac{\partial g}{\partial r} = 0 \tag{23}$$

$$\frac{4m}{(m-2)r}\left\{\frac{\partial f_2}{\partial z} + h\frac{\partial^2 f_1}{\partial z^2} + z\frac{\partial^2 f_2}{\partial z^2}\right\} - \frac{\partial^2 g}{\partial r \partial z} = 0 \tag{24}$$

along the plane $\phi = 0$. Thus, substituting (21) into (22)–(24), one finds respectively:

$$\frac{mh}{m-2}\alpha(\alpha - 1)\{(\alpha - 2)[B_0^{(1)} + B_0^{(2)}]\varrho^{\alpha-3} + (\alpha+1)[B_1^{(1)} + B_1^{(2)}]\varrho^{\alpha-2}\} + 0(\varrho^{\alpha-1}) = 0 \tag{25}$$

$$\frac{2m}{m-2}\alpha(\alpha - 1)\left\{h(\alpha - 2)[A_0^{(1)} + A_0^{(2)}]\varrho^{\alpha-3}\right.$$
$$\left. + \left[h(1 + \alpha)(A_1^{(1)} + A_1^{(2)}) + B_0^{(2)} - \frac{h}{2a}(A_0^{(1)} + A_0^{(2)})\right]\varrho^{\alpha-2}\right\} + 0(\varrho^{\alpha-1}) = 0 \tag{26}$$

and

$$\alpha(1 - \alpha)\left\{\frac{4m}{m-2}\left(\frac{h}{a}\right)(A_0^{(1)} + A_0^{(2)}) - B_0^{(3)}\right\}\varrho^{\alpha-2} + 0(\varrho^{\alpha-1}) = 0, \tag{27}$$

all of which are satisfied if one lets

$$B_0^{(1)} = -B_0^{(2)} \tag{28}$$

$$B_1^{(1)} = -B_1^{(2)} \tag{29}$$

$$A_0^{(1)} = -A_0^{(2)} \tag{30}$$

$$h(1+\alpha)[A_1^{(1)} + A_1^{(2)}] = -B_0^{(2)} \tag{31}$$

$$B_0^{(3)} = 0. \tag{32}$$

Similarly, the boundary conditions along the surface of the hole, i.e. at $\phi = \pi/2$, become:

$$-\frac{2}{m-2}\frac{\partial^2 f_2}{\partial z^2} + \frac{1}{m-2}\frac{\partial^2}{\partial r^2}\left\{2(m-1)f_2 + mh\frac{\partial f_1}{\partial z} + mz\frac{\partial f_2}{\partial z}\right\} - \frac{2}{r}\frac{\partial g}{\partial r} + \frac{2}{r^2}g = 0 \tag{33}$$

$$\frac{2}{m-2}\left\{-\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\right\}\left\{2(m-1)f_2 + mh\frac{\partial f_1}{\partial z} + mz\frac{\partial f_2}{\partial z}\right\} + \frac{\partial^2 g}{\partial r^2} + \frac{1}{2}\frac{\partial^2 g}{\partial z^2} = 0 \tag{34}$$

and

$$\frac{m}{m-2}\frac{\partial}{\partial r}\left\{-\frac{\partial f_2}{\partial z} - h\frac{\partial^2 f_1}{\partial z^2} - z\frac{\partial^2 f_2}{\partial z^2}\right\} + \frac{1}{r}\frac{\partial g}{\partial z} = 0, \tag{35}$$

which, upon the substitution of (21), suggests that the following combinations must vanish

$$\alpha(1-\alpha)\left[\alpha\cos\left(\frac{\alpha\pi}{2}\right)A_0^{(2)} - (1-\alpha)\sin\left(\frac{\alpha\pi}{2}\right)B_0^{(2)}\right]\varrho^{\alpha-2} + 0(\varrho^{\alpha-1}) = 0 \tag{36}$$

$$\alpha(1-\alpha)\cos\left(\frac{\alpha\pi}{2}\right)A_0^{(3)}\varrho^{\alpha-2} + 0(\varrho^{\alpha-1}) = 0 \tag{37}$$

$$\alpha(1-\alpha)\left\{-(1-\alpha)\sin\left(\frac{\alpha\pi}{2}\right)A_0^{(2)} + (2-\alpha)\cos\left(\frac{\alpha\pi}{2}\right)B_0^{(2)}\right\}\varrho^{\alpha-2} + 0(\varrho^{\alpha-1}) = 0 \tag{38}$$

respectively. It is clear, therefore, that

$$A_0^{(3)} = 0 \tag{39}$$

and that the constant α must satisfy the following transcendental equation

$$\alpha^2(1-\alpha)^2\{\cos(\alpha-1)\pi + 2(\alpha-1)^2 - 1\} = 0. \tag{40}$$

An examination of the above equation shows that there exist three real roots, namely $\alpha = 0$, $\alpha = 1$, and $\alpha = 2$, and in addition an infinite number of complex roots with positive real parts. The complex root with the minimum real part is $\alpha = 3.73959 \pm 1.11902i$.

Returning, next, to (14) and (15) one finds, in view of (21), (28)–(32) and (36)–(40), that

$$\sqrt{r} f_2 = B_0^{(2)} \varrho^\alpha \left\{ \frac{1-\alpha}{\alpha} \tan\left(\frac{\alpha\pi}{2}\right) \cos(\alpha\phi) + \sin(\alpha\phi) \right\} \cos(2\theta) + 0(\varrho^{\alpha+1}) \tag{41}$$

$$\sqrt{r}\left\{ h \frac{\partial f_1}{\partial z} + z \frac{\partial f_2}{\partial z} \right\} = B_0^{(2)} \varrho^\alpha \left\{ -\sin(\alpha\phi) + \alpha \sin\phi \left[\cos(1-\alpha)\phi + \frac{1-\alpha}{\alpha} \tan\left(\frac{\alpha\pi}{2}\right) \sin(1-\alpha)\phi \right] \right\} \cos(2\theta) + 0(\varrho^{\alpha+1}). \tag{42}$$

$$\sqrt{r} g = 0 + 0(\varrho^{\alpha+1}), \tag{43}$$

from which the displacement and stress fields can now be obtained. Moreover, in view of (20), the entire solution may easily be constructed in ascending powers of ϱ. Without going into the mathematical details, one finds

(i) *the displacement field*:

$$u = \frac{m}{m-2} \varrho^{\alpha-1} \Psi(\phi) \sin(\theta) \cos(2\theta) + 0(\varrho^\alpha) \tag{44}$$

$$v = \frac{m}{m-2} \varrho^{\alpha-1} \Psi(\phi) \cos(\theta) \cos(2\theta) + 0(\varrho^\alpha) \tag{45}$$

$$w = -\frac{m}{m-2} \varrho^{\alpha-1} \Phi(\phi) \cos(2\theta)\, 0(\varrho^\alpha) \tag{46}$$

where

$$\Psi(\phi) = \alpha B_0^{(2)} \left\{ 2\left(\frac{m-1}{m-2}\right)\left(\frac{1-\alpha}{\alpha}\right) \tan\left(\frac{\alpha\pi}{2}\right) \cos(\alpha-1)\phi + \frac{m-2}{m} \sin(\alpha-1)\phi \right\} + \alpha(\alpha-1) B_0^{(2)} \sin\phi \left\{ \frac{1-\alpha}{\alpha} \tan\left(\frac{\alpha\pi}{2}\right) \sin(2-\alpha)\phi + \cos(2-\alpha)\phi \right\} \tag{47}$$

$$\Phi(\phi) = \alpha B_0^{(2)} \left\{ \frac{3m-2}{m}\left(\frac{1-\alpha}{\alpha}\right) \tan\left(\frac{\alpha\pi}{2}\right) \sin(\alpha-1)\phi - 2\left(\frac{m-1}{m}\right) \cos(\alpha-1)\phi \right\} + \alpha(1-\alpha) B_0^{(2)} \sin\phi \left\{ \frac{1-\alpha}{\alpha} \tan\left(\frac{\alpha\pi}{2}\right) \cos(\alpha-2)\phi + \sin(\alpha-2)\phi \right\}. \tag{48}$$

(ii) *the stress field*:

$$\sigma_{zz} = \frac{2mG}{m-2}\alpha(\alpha-1)\varrho^{\alpha-2}B_0^{(2)}\left\{\sin(\alpha-2)\phi + (\alpha-2)\sin\phi \cdot \left[\left(\frac{1-\alpha}{\alpha}\right)\tan\left(\frac{\alpha\pi}{2}\right)\sin(\alpha-3)\phi - \cos(\alpha-3)\phi\right]\right\}\cos(2\theta) + 0(\varrho^{\alpha-1}). \tag{49}$$

$$\tau_{rz} = \frac{2mG}{m-2}\alpha(\alpha-1)\varrho^{\alpha-2}B_0^{(2)}\left\{\left(\frac{1-\alpha}{\alpha}\right)\tan\left(\frac{\alpha\pi}{2}\right)\sin(\alpha-2)\phi + (\alpha-2)\sin\phi \cdot \left[\left(\frac{1-\alpha}{\alpha}\right)\tan\left(\frac{\alpha\pi}{2}\right)\cos(\alpha-3)\phi + \sin(\alpha-3)\phi\right]\right\}\cos(2\theta) + 0(\varrho^{\alpha-1}) \tag{50}$$

$$\tau_{xz} = \tau_{rz}\sin\theta + 0(\varrho^{\alpha-1}) \tag{51}$$

$$\tau_{yz} = \tau_{rz}\cos\theta + 0(\varrho^{\alpha-1}) \tag{52}$$

$$\tau_{r\theta} = 0(\varrho^{\alpha-1}) \tag{53}$$

$$\sigma_{rr} = \frac{2mG}{m-2}\alpha(\alpha-1)\,\varrho^{\alpha-2}B_0^{(2)}\left\{2\left(\frac{1-\alpha}{\alpha}\right)\tan\left(\frac{\alpha\pi}{2}\right)\cos(\alpha-2)\phi + \sin(\alpha-2)\phi - (\alpha-2)\sin\phi\left[\left(\frac{1-\alpha}{\alpha}\right)\tan\left(\frac{\alpha\pi}{2}\right)\sin(\alpha-3)\phi - \cos(\alpha-3)\phi\right]\right\} \times \cos(2\theta) + 0(\varrho^{\alpha-1}) \tag{54}$$

$$\sigma_{\theta\theta} = -\frac{2mG}{m-2}\alpha(\alpha-1)\,\varrho^{\alpha-2}B_0^{(2)}\left\{2\left(\frac{m+1}{m}\right)\left(\frac{1-\alpha}{\alpha}\right)\tan\left(\frac{\alpha\pi}{2}\right)\cos(\alpha-2)\phi + \left(\frac{m+2}{m}\right)\sin(\alpha-2)\phi - (\alpha-2)\sin\phi\left[\left(\frac{1-\alpha}{\alpha}\right)\tan\left(\frac{\alpha\pi}{2}\right)\sin(\alpha-3)\phi - \cos(\alpha-3)\phi\right]\right\}\cos(2\theta) + 0(\varrho^{\alpha-1}) \tag{55}$$

Finally, it should be noted that the coefficient $[B_0^{(2)}/m - 2]$ can be shown to be proportional to Poisson's ratio ν. This matter has been discussed in [5] where the solution, throughout the thickness of the plate, is presented.

Conclusions

The foregoing analysis clearly shows that the stress field in the neighborhood of the corner point, i.e., the point where the hole surface intersects with the free of stress boundary plane, has no stress singularities present, provided that the radius of the hole is sufficiently large that the condition $\varrho \ll a$ is meaningful. Moreover, the stress field has been shown to be proportional to $\varrho^{\alpha-2}$. It is interesting to note that the first root is the same as that obtained by the William's solution for a 90 deg corner with free–free of stress boundaries,* i.e., $\alpha = 3.73959 \pm i1.11902$. While this result was to be anticipated on intuitive grounds, it could not be taken for granted.

An extension of this analysis to other angles of intersection reveals the same first root as that predicted by the William's solution. Thus the William's solution has a further applicability than was originally thought.

Perhaps it is noteworthy to point out the importance of the general 3D solution for it reveals the inherent form of the solution at such neighborhoods thus making its explicit construction possible. This simple problem was used as a vehicle to show how some of these characteristics may be utilized in order to construct such a solution. For the solution in the interior of the plate, the reader is referred to [5].

References

1. M.L. Williams, *Journal of Applied Mechanics* 19 (1952) 526–528.
2. M.L. Williams, "Surface Singularities Resulting from Various Boundary Conditions in Angular Corners of Plates under Bending." U.S. National Congress of Applied Mechanics, Illinois Institute of Technology, Chicago, Ill. (June 1951).
3. W.S. Burton, G.B. Sinclair, J.S. Solecki and J.L. Swedlow, *International Journal of Fracture* 25 (1984).
4. E.S. Folias, *Journal of Applied Mechanics* 42 (1975) 663–674.
5. E.S. Folias and J.J. Wang, "On the Three-Dimensional Stress Field Around a Circular Hole in a Plate of Arbitrary Thickness," UTEC Report University of Utah (May 1986).
6. O.K. Aksentian, *PMM* 31 (1967).

Résumé. Dans le mémoire, l'auteur étudie le champ analytique des contraintes au voisinage de l'intersection d'un trou et d'une surface libre. En utilisant une forme de solution générale à trois dimensions, on retrouve les champs explicites de déplacement et de contraintes en une telle zone. De plus, l'analyse montre que la solution complémentaire doit être proportionnelle à ϱ exp ($\alpha - 2$), où α est indépendante du module de Poisson ν. Il est intéressant de noter que la racine premiére obtenue est exactement celle que fournit la solution par Williams du problème du coin à 90° dont les surfaces libres sont libres de contraintes.

*Prof. G. Sinclair recently brought to the attention of the author the work of Aksetian [6], who using the method of scaling was able to obtain the same characteristic value without actually having to solve the boundary value problem.