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On the Corner Singularity of a 3-D Griffith Crack

by

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Abstract

This report discusses some further developments of an analytical solution to the 3-D Griffith crack problem. The analysis shows the stresses at the corner points to be singular of the order $(1/2 + 2\nu)$. Moreover, the stress boundary conditions at the plate faces are shown to be proportional to $(h - z)$, at the upper face, and to $(h + z)$, at the lower face.

Introduction

A major debility in current fracture mechanics work is the ignorance of the effects of thickness on the mechanism of failure. For example, the common experimental observation of a change from ductile failure at the edge to brittle fracture at the center of a broken sheet material has so far defied analysis. Moreover, fracture toughness of a material is presently assumed to be constant. Yet engineering tables prescribe two different values for thin and thick plates. More frustrating yet is the fact that there exists no precise definition as to what represents a 'thin' or a 'thick' plate. Obviously, the concept of fracture toughness, as it is presently defined, is an inappropriate parameter to use for the prediction of failures due to fracture. Yet an orderly theoretical attack on the problem can provide important guidance to these and other phases of fracture research. The most potent mathematical tool for this attack is the linear theory of infinitesimal elasticity as applied to a cracked plate of finite thickness. Although this theory cannot include the nonelastic behavior of the material at the crack tip per se, it can evince many characteristics of the actual behavior of a cracked plate, including those due to thickness.

For example, such information could be valuable to the understanding and the solution of the corresponding 3-D elastoplastic problem. While it is true that this theory cannot give us the exact stress state at a point in the interior of the shear lip, it can, however, prescribe fairly accurately the shear lip envelope. Thus the theory of linear elasticity is a logical fountainhead for detailed theoretical study.*

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*It is for these reasons that a National Workshop on Three-Dimensional Fracture Analysis, held at Battelle on April 26 - 28, 1976, identified this as one of the Benchmark problems in the field of Fracture Mechanics.

The mathematical difficulties, however, posed by three-dimensional crack problems are substantially greater than those associated with plane stress or plane strain. As a result, there exist in the literature very few analytical papers that deal specifically with the three-dimensional stress character at the base of a stationary crack.* Simultaneously, in the last two decades, numerous attempts have been made to obtain a finite element solution by very capable researchers. Unfortunately, they too experienced difficulties and their respective results were contradictory.**

Review of author's past work.

In 1973, Folias, using a method developed by the Russian elastician Lur'e [1] and the application of Fourier Integral Transforms, constructed a solution [2] to Navier's equations for a mixed boundary value problem, that of a 3-D Griffith crack (see fig. 1). The integral representations were subsequently expanded asymptotically in the inner layers of the plate and the displacement and stress fields were found to be:

(i) Displacements:

$$u^{(c)} = \bar{\sigma}_0 \frac{\Lambda}{2G} \left\{ \frac{1}{\left(1 - \frac{z}{h}\right)^{2\nu}} + \frac{1}{\left(1 + \frac{z}{h}\right)^{2\nu}} \right\} \sqrt{\frac{c\varepsilon}{2}} \left\{ \frac{m - 2}{m} \cos\left(\frac{\phi}{2}\right) + \frac{1}{2} \sin\phi \sin\left(\frac{\phi}{2}\right) \right\} + O(\varepsilon') \qquad (1)$$

*For a historical discussion see reference [3].

**Refer to proceedings of Workshop on Three-Dimensional Fracture Analysis held at Battelle, 1976.

$$v^{(c)} = \bar{\sigma}_0 \frac{\Lambda}{2G} \{ \frac{1}{(1-\frac{z}{h})^{2\nu}} + \frac{1}{(1+\frac{z}{h})^{2\nu}} \} \sqrt{\frac{c\varepsilon}{2}} \{ 2 (\frac{m-1}{m}) \sin (\frac{\phi}{2})$$

$$- \frac{1}{2} \sin \phi \cos (\frac{\phi}{2}) \} + 0(\varepsilon') \qquad (2)$$

$$w^{(c)} = 0 + 0(\varepsilon') \qquad (3)$$

(ii) Stresses:

$$\sigma_x{}^{(c)} = \bar{\sigma}_0 \Lambda \{ \frac{1}{(1-\frac{z}{h})^{2\nu}} + \frac{1}{(1+\frac{z}{h})^{2\nu}} \} \sqrt{\frac{c}{2\varepsilon}} \{ \frac{1}{2} \cos (\frac{\phi}{2})$$

$$- \frac{1}{4} \sin \phi \sin (\frac{3\phi}{2}) \} + 0(\varepsilon^0) \qquad (4)$$

$$\sigma_y{}^{(c)} = \bar{\sigma}_0 \Lambda \{ \frac{1}{(1-\frac{z}{h})^{2\nu}} + \frac{1}{(1+\frac{z}{h})^{2\nu}} \} \sqrt{\frac{c}{2\varepsilon}} \{ \frac{1}{2} \cos (\frac{\phi}{2})$$

$$+ \frac{1}{4} \sin \phi \sin (\frac{3\phi}{2}) \} + 0(\varepsilon^0) \qquad (5)$$

$$\sigma_z{}^{(c)} = \nu \bar{\sigma}_0 \Lambda \{ \frac{1}{(1-\frac{z}{h})^{2\nu}} + \frac{1}{(1+\frac{z}{h})^{2\nu}} \} \sqrt{\frac{c}{2\varepsilon}} \cos (\frac{\phi}{2}) + 0(\varepsilon^0) \qquad (6)$$

$$\tau_{xy}{}^{(c)} = \bar{\sigma}_0 \Lambda \{ \frac{1}{(1-\frac{z}{h})^{2\nu}} + \frac{1}{(1+\frac{z}{h})^{2\nu}} \} \sqrt{\frac{c}{2\varepsilon}} \{ \frac{1}{4} \sin \phi \cos (\frac{3\phi}{2}) \}$$

$$+ 0(\varepsilon^0) \qquad (7)$$

$$\tau_{yz}^{(c)} = -\nu\bar{\sigma}_0 \frac{\Lambda}{h} \left\{ \frac{1}{(1-\frac{z}{h})^{2\nu+1}} - \frac{1}{(1+\frac{z}{h})^{2\nu+1}} \right\} \cdot$$

$$\sqrt{\frac{c\varepsilon}{2}} \left\{ \frac{1}{2} \sin\phi \cos\left(\frac{\phi}{2}\right)\right\} + 0(\varepsilon') \tag{8}$$

$$\tau_{xz}^{(c)} = \nu\bar{\sigma}_0 \frac{\Lambda}{h} \left\{ \frac{1}{(1-\frac{z}{h})^{2\nu+1}} - \frac{1}{(1+\frac{z}{h})^{2\nu+1}} \right\} \cdot$$

$$\sqrt{\frac{c\varepsilon}{2}} \left\{ (1-2\nu)\cos\left(\frac{\phi}{2}\right) + \frac{1}{2}\sin\phi\sin\left(\frac{\phi}{2}\right)\right\} + 0(\varepsilon'), \tag{9}$$

where Λ is a function of Poisson's ratio ν and c/h.

The above solution reveals the following important characteristics, which are applicable only in the inner layers of the plate:

(1) the stresses possess the usual $1/\sqrt{\varepsilon}$ singularity,

(2) the stresses possess the usual angular distribution,

(3) the stress intensity factor K_I is a function of z,

(4) exact plane strain conditions exist only on the plane $z = 0$,

(5) a pseudo plane strain state* exists and the equation

$$\sigma_z = \nu(\sigma_x + \sigma_y) \tag{10}$$

(6) as the plate thickness $2h \rightarrow \infty$, the plane strain solution is recovered,

(7) as Poisson's ratio $\nu \rightarrow 0$, the plane stress solution is recovered,

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*The reader should note that the stresses are functions of z, therefore the term 'pseudo' plane strain.

and also serve as check points for the validity of the solution, at least as $\nu \longrightarrow 0$ and as $h \longrightarrow \infty$.

One of the great advantages of an analytical expression is that it can easily be verified as to whether it represents a solution to a problem by direct substitution into the governing equations and the appropriate boundary conditions. The reader should note that the above displacement field satisfies all Navier's equations and that the stress field satisfies the following boundary conditions

$$\tau_{xy}^{(c)} = \tau_{yz}^{(c)} = 0 \quad ; \quad \text{for } \phi = 0 \text{ and } \pi$$

and

$$\sigma_{y}^{(c)} = 0 \quad ; \quad \text{for } \phi = \pi .$$

Inasmuch as the solution represents an asymptotic expansion which is valid only in the inner layers of the plate, naturally the stress field can not be expected to satisfy the boundary conditions on the plate faces, i.e. at $z = \pm h$. The author, however, shows* in reference [2] that in the neighborhood of the corner point additional terms** also contribute to the same order of singularity and therefore must be accounted for.*** Moreover, it was shown that all stresses there, inclusive of $\tau_{xz}^{(c)}$ and $\tau_{yz}^{(c)}$, possess the same order of singularity. He, subsequently, ventured to examine the character of the stress singularity which prevails at the corner point, i.e. the point where the crack front meets the free surface of the plate. After

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*See pages 669 - 671.

**This matter will also be addressed at a later time.

***It should be pointed out that the integral representations do satisfy the boundary conditions there and that this comment refers only to the recovery of the complete local expansion from the integral representations.

considerable effort, it was concluded that the displacements are proportional to $\rho^{1/2-2\nu}$ and similarly the stresses are proportional to $\rho^{-1/2-2\nu}$.

Researchers in the field of fracture mechanics, however, were reluctant to accept at that time the possibility of an infinite displacement field*, on the basis of physical intuition. Consequently, the results were considered to be controversial and the following legitimate and fundamental questions were raised:

1. Is the solution of this notorious difficult problem unique and if so under what conditions?
2. Is an infinite displacement field, in this case, admissible?
3. Does the symbolic method adopted by the author[2] generate a 'complete' set of eigenfunctions for the solution of Navier's equations?

The answers to the above questions were given by Prof. Calvin Wilcox.**

First of all, he was successful in proving [4] that a displacement field which satisfies the condition of local finite energy is unique. This of course is quite a departure from our traditional 2-D fracture mechanics thinking, for the displacements may now be allowed to be singular. Consequently, one may not a priori assume them to be finite as it is customarily done. In general, such an assumption makes the class of solutions too

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*Infinite displacement fields are not uncommon in the theory of linear elasticity. For example, it is well established that the Boussinesq's, Kelvin's and Cerruti's problems have displacement fields that behave like $u_i \sim \rho^{-1}$. The reader should also be reminded that the reported results of the crack problem correspond to the complementary solution, i.e. the interior of the crack faces is subjected to a uniform fluid pressure.

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restrictive and, as a result, one may not find the complete solution to the problem. On the other hand, the solution could very well give finite displacements everywhere! Be that as it may, physical intuition should be used with extreme caution.

Second, he was able to show [5] that the Fourier integral expressions* representing the general solution to Navier's equations are complete and, furthermore, the 'symbolic method' used is justifiable. In order to prove this, he used a double Fourier integral transform in x and y and subsequently a contour integration to recover precisely the same expressions as those reported by Folias in [2].

Finally, it remains to determine explicitly the stress field ahead of the crack tip and throughout the thickness of the plate. Moreover, it would be interesting to see if the asymptotic expansion in the inner layers of the plate matches the stress field given by equations (4) - (9).

The Complementary Displacement and Stress Fields

From reference [3], we have the following complementary displacement and stress fields.

(1) the displacement field:

$$
\begin{aligned}
u^{(c)} = {} & \frac{1}{m-2} \sum_{\nu=1}^{\infty} A_\nu \frac{\partial^2 H_\nu}{\partial x^2} \{ 2(m-1) \cos(\beta_\nu h) \cos(\beta_\nu z) \\
& + m \beta_\nu h \sin(\beta_\nu h) \cos(\beta_\nu z) - m \beta_\nu z \cos(\beta_\nu h) \sin(\beta_\nu z) \} \\
& + \sum_{n=1}^{\infty} B_n \{ - \frac{\partial^2 H_n^*}{\partial x^2} + \alpha_n^2 H_n^* \} \cos(\alpha_n h) \cos(\alpha_n z) \\
& + I_1 - |y| \frac{\partial I_2}{\partial x} + \frac{1}{m+1} z^2 \frac{\partial^2 I_3}{\partial x \partial |y|}
\end{aligned}
\tag{11}
$$

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*See Eqns. (52) - (54) of [4].

$$v^{(c)} = \frac{1}{m-2} \sum_{\nu=1}^{\infty} A_\nu \frac{\partial^2 H_\nu}{\partial x \partial y} \{ 2(m-1) \cos(\beta_\nu h) \cos(\beta_\nu z)$$

$$+ m \beta_\nu h \sin(\beta_\nu h) \cos(\beta_\nu z) - m \beta_\nu z \cos(\beta_\nu h) \sin(\beta_\nu z) \}$$

$$- \sum_{n=1}^{\infty} B_n \frac{\partial^2 H_n^*}{\partial x \partial y} \cos(\alpha_n h) \cos(\alpha_n z) - \frac{3m-1}{m+1} I_3 - I_2 \tag{12}$$

$$+ |y| \frac{\partial I_3}{\partial |y|} + \frac{1}{m+1} z^2 \frac{\partial^2 I_3}{\partial x^2}$$

$$w^{(c)} = \frac{1}{m-2} \sum_{\nu=1}^{\infty} A_\nu \frac{\partial H_\nu}{\partial x} \beta_\nu \{ (m-2) \cos(\beta_\nu h) \sin(\beta_\nu z)$$

$$- m \beta_\nu h \sin(\beta_\nu h) \sin(\beta_\nu z) - m \beta_\nu z \cos(\beta_\nu h) \cos(\beta_\nu z) \} \tag{13}$$

$$- \frac{2}{m+1} z \frac{\partial I_3}{\partial |y|}$$

where for simplicity we have adopted the following definitions:

$$A_\nu \cdot H_\nu (x, y; \beta_\nu) = \int_0^\infty \frac{\Gamma_\nu}{s^2} \frac{e^{-\sqrt{s^2 + \beta_\nu^2}\,|y|}}{\sqrt{s^2 + \beta_\nu^2}} \sin(sx)\, ds \,, \tag{14}$$

$$(-1)^n B_n H_n^* (x, y; \alpha_n) = \int_0^\infty \frac{s_n}{\sqrt{s^2 + \alpha_n^2}} \frac{e^{-\sqrt{s^2 + \alpha_n^2}\,|y|}}{\sqrt{s^2 + \alpha_n^2}} \sin(sx)\, ds \,, \tag{15}$$

with A_ν and B_n as functions of β_ν and α_n respectively,

$$\alpha_n \equiv \frac{n\pi}{h} \qquad n = 1, 2, 3, \ldots \,, \tag{16}$$

β_ν = the roots of the transidental equation

$$\sin(2\beta_\nu h) = -2(\beta_\nu h) \quad , \tag{17}$$

and

$$I_1 = \int_0^\infty P(s)\, e^{-s|y|} \sin(sx)\, ds \tag{18}$$

$$I_2 = \int_0^\infty P(s)\, e^{-s|y|} \cos(sx)\, ds \tag{19}$$

$$I_3 = \int_0^\infty \frac{Q(s)}{s}\, e^{-s|y|} \cos(sx)\, ds \quad . \tag{20}$$

(ii) <u>the stress field</u>:

$$\frac{1}{2G}\sigma_x^{(c)} = \frac{2}{m-2}\sum_{\nu=1}^{\infty} A_\nu \beta_\nu^2 \frac{\partial H_\nu}{\partial x} \cos(\beta_\nu h)\cos(\beta_\nu z)$$

$$+ \frac{1}{m-2}\sum_{\nu=1}^{\infty} A_\nu \frac{\partial^3 H_\nu}{\partial x^3}\{2(m-1)\cos(\beta_\nu h)\cos(\beta_\nu z)$$

$$+ m\,\beta_\nu h\sin(\beta_\nu h)\cos(\beta_\nu z) - m\,\beta_\nu z\cos(\beta_\nu h)\sin(\beta_\nu z)\} \tag{21}$$

$$+ \sum_{n=1}^{\infty} B_n \{-\frac{\partial^3 H_n^*}{\partial x^3} + \alpha_n^2 \frac{\partial H_n^*}{\partial x}\}\cos(\alpha_n h)\cos(\alpha_n z)$$

$$+ \frac{\partial I_1}{\partial x} - |y|\frac{\partial^2 I_3}{\partial x^2} + \frac{1}{m+1} z^2 \frac{\partial^2 I_2}{\partial x^2\,\partial|y|} + \frac{2}{m+1}\frac{\partial I_3}{\partial|y|}$$

$$\frac{1}{2G}\sigma_y^{(c)} = \frac{2}{m-2}\sum_{\nu=1}^{\infty}\beta_\nu^{\,2} A_\nu \frac{\partial H_\nu}{\partial x}\cos(\beta_\nu h)\cos(\beta_\nu z)$$

$$-\frac{1}{m-2}\sum_{\nu=1}^{\infty} A_\nu\left\{\frac{\partial^3 H_\nu}{\partial x^3} - \beta_\nu^{\,2}\frac{\partial H_\nu}{\partial x}\right\}\{2(m-1)\cos(\beta_\nu h)\cos(\beta_\nu z)$$

$$+ m\,\beta_\nu h\sin(\beta_\nu h)\cos(\beta_\nu z) - m\beta_\nu z\cos(\beta_\nu h)\sin(\beta_\nu z)\} \quad (22)$$

$$+\sum_{n=1}^{\infty} B_n\left\{\frac{\partial^3 H_n^*}{\partial x^3} - \alpha_n^{\,2}\frac{\partial H_n^*}{\partial x}\right\}\cos(\alpha_n h)\cos(\alpha_n z)$$

$$+\frac{2m}{m+1}\frac{\partial I_3}{\partial|y|} + \frac{\partial I_1}{\partial x} + |y|\frac{\partial^2 I_3}{\partial x^2} - \frac{1}{m+1}z^2\frac{\partial^3 I_3}{\partial x^2\,\partial|y|}$$

$$\frac{1}{2G}\sigma_z^{(c)} = \frac{m}{m-2}\sum_{\nu=1}^{\infty} A_\nu\frac{\partial H_\nu}{\partial x}\beta_\nu^{\,2}\{-\beta_\nu h\sin(\beta_\nu h)\cos(\beta_\nu z)$$

$$(23)$$

$$+\beta_\nu z\cos(\beta_\nu h)\sin(\beta_\nu z)\}$$

$$\frac{1}{G}\tau_{xy}^{(c)} = \frac{2}{m-2}\sum_{\nu=1}^{\infty} A_\nu\frac{\partial^3 H_\nu}{\partial x^2\partial y}\{2(m-1)\cos(\beta_\nu h)\cos(\beta_\nu z)$$

$$+ m\beta_\nu h\sin(\beta_\nu h)\cos(\beta_\nu z) - m\beta_\nu z\cos(\beta_\nu h)\sin(\beta_\nu z)\}$$

$$(24)$$

$$-\sum_{n=1}^{\infty} B_n\left\{2\frac{\partial^3 H_n^*}{\partial x^2\partial y} - \alpha^{\,2}\frac{\partial H_n^*}{\partial y}\right\}\cos(\alpha_n h)\cos(\alpha_n z)$$

$$-2\left(\frac{m-1}{m+1}\right)\frac{\partial I_3}{\partial x} - 2\frac{\partial I_2}{\partial x} + \frac{2}{m+1}z^2\frac{\partial^3 I_3}{\partial x^3} + 2|y|\frac{\partial^2 I_3}{\partial x\,\partial|y|}$$

$$\frac{1}{G}\tau_{xz}^{(c)} = -\frac{2m}{m-2}\sum_{\nu=1}^{\infty} A_\nu \frac{\partial^2 H_\nu}{\partial x^2}\beta_\nu \{\cos(\beta_\nu h)\sin(\beta_\nu z)$$

$$+\beta_\nu h\sin(\beta_\nu h)\sin(\beta_\nu z)+\beta_\nu z\cos(\beta_\nu h)\cos(\beta_\nu z)\} \qquad (25)$$

$$+\sum_{n=1}^{\infty} B_n\{\frac{\partial^2 H_n^*}{\partial x^2}-\alpha_n^2 H_n^*\}\alpha_n\cos(\alpha_n h)\sin(\alpha_n z)$$

$$\frac{1}{G}\tau_{yz}^{(c)} = -\frac{2m}{m-2}\sum_{\nu=1}^{\infty} A_\nu\beta_\nu\frac{\partial^2 H_\nu}{\partial x\partial y}\{\cos(\beta_\nu h)\sin(\beta_\nu z)$$

$$+\beta_\nu h\sin(\beta_\nu h)\sin(\beta_\nu z)+\beta_\nu z\cos(\beta_\nu h)\cos(\beta_\nu z)\} \qquad (26)$$

$$+\sum_{n=1}^{\infty} B_n\alpha_n\frac{\partial^2 H_n^*}{\partial x\partial y}\cos(\alpha_n h)\sin(\alpha_n z)$$

By direct substitution, it can easily be ascertained that the foregoing complementary displacement field satisfies Navier's equations and furthermore the corresponding stresses $\sigma_z^{(c)}$, $\tau_{zx}^{(c)}$, $\tau_{yz}^{(c)}$ do vanish at the plate faces $z = \pm h$. The remaining boundary conditions:

$$\tau_{yz}^{(c)} = \tau_{yx}^{(c)} = 0 \;; \quad \text{for} \quad y = 0 \text{ and all } x \qquad (27)-(28)$$

and

$$\sigma_y^{(c)} = -\sigma_0 \;; \quad \text{for} \quad y = 0 \text{ and all } x < c, \qquad (29)$$

are satisfied, if one considers the following combinations to vanish respectively:

$$- \frac{2m}{m-2} \sum_{\nu=1}^{\infty} A_\nu \left(\frac{\partial H_\nu}{\partial y}\right)_{y=0} \{ \beta_\nu h \sin(\beta_\nu h) \cos(\beta_\nu z)$$

$$- \beta_\nu z \cos(\beta_\nu h) \sin(\beta_\nu z) \} + \sum_{n=1}^{\infty} B_n \left(\frac{\partial H_n}{\partial y}\right)_{y=0}$$

$$\cos(\alpha_n h) \cos(\alpha_n z) + \frac{4m}{m-2} \sum_{\nu=1}^{\infty} A_\nu \left(\frac{\partial H_\nu}{\partial y}\right)_{y=0} = 0 ; \tag{30}$$

for all x .

$$\frac{2}{m-2} \sum_{\nu=1}^{\infty} A_\nu \left(\frac{\partial^3 H_\nu}{\partial x^2 \partial y}\right)_{y=0} \{ 2(m-1) \cos(\beta_\nu h) \cos(\beta_\nu z)$$

$$+ m \beta_\nu h \sin(\beta_\nu h) \cos(\beta_\nu z) - m \beta_\nu z \cos(\beta_\nu z) \sin(\beta_\nu z) \}$$

$$- \sum_{n=1}^{\infty} B_n \{ 2 \frac{\partial^3 H_n^*}{\partial x^2 \partial y} - \alpha_n^2 \frac{\partial H_n^*}{\partial y} \}_{y=0} \cos(\alpha_n h) \cos(\alpha_n z)$$

$$+ \{ -2 \left(\frac{m-1}{m+1}\right) \frac{\partial I_3}{\partial x} - 2 \frac{\partial I_2}{\partial x} + \frac{2}{m+1} z^2 \frac{\partial^3 I_3}{\partial x^3} \}_{y=0} = 0 \tag{31}$$

for all x ,

and

$$\frac{2}{m-2} \sum_{\nu=1}^{\infty} \beta_\nu^2 A_\nu \left(\frac{\partial H_\nu}{\partial x}\right)_{y=0} \cos(\beta_\nu h) \cos(\beta_\nu z)$$

$$- \frac{1}{m-2} \sum_{\nu=1}^{\infty} A_\nu \{ \frac{\partial^3 H_\nu}{\partial x^3} - \beta_\nu^2 \frac{\partial H_\nu}{\partial x} \}_{y=0} \{ 2(m-1) \cos(\beta_\nu h) \cdot$$

$$\cos(\beta_\nu z) + m\beta_\nu h \sin(\beta_\nu h)\cos(\beta_\nu z) - m\,\beta_\nu z \,\cdot$$

$$\cos(\beta_\nu h)\sin(\beta_\nu z)\,\} \;+\; \sum_{n=1}^{\infty} B_n \left\{ \frac{\partial^3 H_n^*}{\partial x^3} - \alpha_n^2 \frac{\partial H_n^*}{\partial x} \right\}_{y=0} \cdot \tag{32}$$

$$\cos(\alpha_n h)\cos(\alpha_n z) + \left\{ \frac{2m}{m+1}\frac{\partial I_3}{\partial |y|} + \frac{\partial I_1}{\partial x} \right.$$

$$\left. - \frac{1}{m+1} z^2 \frac{\partial^3 I_3}{\partial x^2 \partial |y|} \right\}_{y=0} = -\left(\frac{\sigma_0}{2G}\right) \; ; \quad |x| < c \; .$$

Next, upon utilizing eq. (30) into eq. (31), one finds: after some rearrangement:

$$\frac{\partial^2}{\partial x^2}\left\{ \frac{2}{m-2}\sum_{\nu=1}^{\infty} A_\nu \left(\frac{\partial H_\nu}{\partial y}\right)_{y=0} [\, 2(m-1)\cos(\beta_\nu h)\cos(\beta_\nu z) \right.$$

$$- m\,\beta_\nu h \sin(\beta_\nu h)\cos(\beta_\nu z) + m\,\beta_\nu z\,\cos(\beta_\nu h)\sin(\beta_\nu z)\,]$$

$$\left. + \frac{8m}{m-2}\sum_{\nu=1}^{\infty} A_\nu \left(\frac{\partial H_\nu}{\partial y}\right)_{y=0} \right\}$$

$$- \frac{\partial^2}{\partial z^2}\; \frac{2m}{m-2}\sum_{\nu=1}^{\infty} A_\nu \left(\frac{\partial H_\nu}{\partial y}\right)_{y=0} [\, \beta_\nu h \sin(\beta_\nu h)\cos(\beta_\nu z) \tag{33}$$

$$- \beta_\nu z \cos(\beta_\nu h)\sin(\beta_\nu z)\,]\,\}$$

$$= -\left\{ -2\left(\frac{m-1}{m+1}\right)\frac{\partial I_3}{\partial x} - 2\frac{\partial I_2}{\partial x} + \frac{2}{m+1} z^2 \frac{\partial^3 I_3}{\partial x^3} \right\}_{y=0} \; ;$$

for all x .

The problem, therefore, has been reduced to that of solving equations (30), (32) and (33) for the complex coefficients A_ν, the real coefficients B_n, and the respective functions $H_\nu(x,y;\beta_\nu)$ and $H_n^*(x,y;\alpha_n)$ subject to the continuity conditions

$$\left.\frac{\partial H_\nu}{\partial y}\right|_{y=0} = 0 \quad ; \quad |x| > c \tag{34}$$

$$\left.\frac{\partial H_n^*}{\partial y}\right|_{y=0} = 0 \quad ; \quad |x| > c \, . \tag{35}$$

However, before one engages into a lengthy numerical procedure for the solution of the system, it would be desirable to construct, if possible, the functions $H_\nu(x,y;\beta_\nu)$ and $H_n^*(x,y;\alpha_n)$ analytically. After considerable mathematical manipulations it is possible to construct these functions. For example, without going into the mathematical details*, H_n^* is found to be:

$$H_n^*(x,y;\alpha_n) = -\left\{ x\, e^{-\alpha_n|y|} - \frac{1}{2\sqrt{\pi\alpha_n}}\left[\sqrt{r_1 + |y|} \mp \sqrt{r_1 - |y|}\right] e^{-\alpha_n r_1} + \frac{1}{2\sqrt{\pi\alpha_n}}\left[\sqrt{r_2 + |y|} - \sqrt{r_2 - |y|}\right] e^{-\alpha_n r_2} \right\}$$

$$\mp \frac{1}{2}\left(x - \frac{1}{2\alpha_n}\right)\left\{ e^{-\alpha_n|y|}\,\mathrm{Erf}\left[\sqrt{\alpha_n(r_1 - |y|)}\right] \right. \tag{36}$$

$$\mp e^{\alpha_n|y|}\,\mathrm{Erf}\left[\sqrt{\alpha_n(r_1 + |y|)}\right] + \frac{1}{2}\left(x + \frac{1}{2\alpha_n}\right) \cdot$$

$$\left[e^{-\alpha_n|y|}\,\mathrm{Erf}\left[\sqrt{\alpha_n(r_2 - |y|)}\right] - e^{\alpha_n|y|}\,\mathrm{Erf}\left[\sqrt{\alpha_n(r_2 + |y|)}\right]\right]\}$$

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*See footnote on next page.

where we have adopted the following definitions

$$r_1 = \sqrt{(x-c)^2 + y^2} \tag{37a}$$

$$r_2 = \sqrt{(x+c)^2 + y^2} \tag{37b}$$

$$\mathrm{Erf}\,(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\eta^2}\, d\eta \ , \tag{38}$$

and the upper and lower signs refer to $|x| < c$ and $|x| > c$, respectively.

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*For the sake of brevity and in order not to lose the reader with the long and tedious manipulations we omit this derivation. The reader will find it easier to show that this expression satisfies the differential equation:

$$\frac{\partial^2 H_n^*}{\partial x^2} + \frac{\partial^2 H_n^*}{\partial y^2} - \alpha_n^2 H_n^* = 0,$$

the continuity condition (35), the shear boundary conditions $\tau_{yx}^{(c)}$ and $\tau_{yz}^{(c)}$ at $y = 0$ and for $|x| > c$ and furthermore along the crack faces

$$\left.\frac{\partial H_n^*}{\partial x}\right|_{y=0} = -1 .$$

A similar expression may be obtained for $H_\nu(x,y;\beta_\nu)$ as a combination of the function (36) and its derivatives. This step was very important for it enables one to extract the dependence in x and y analytically rather than numerically. In summary, therefore, the problem has now been reduced to that of solving equations (30), (32) and (33) for the complex constant coefficients A_ν and the real but constant coefficients B_n. Moreover, the displacement and stress fields may now be expressed in terms of one summation rather than one summation and one integral.

Although the numerical solution of these coefficients is now possible, from which the singularity may then be determined (at least in principle), we believe that such procedure may not lead us to all the singularities of the problem* and that such information should be extracted by analytical means.

The behavior of the solution at the corner.

One of the acadamic questions that has defied researchers for a long time is the order of the singularity that prevails at the corner point. Thus, in order for us to study the stress distribution in the vicinity of that point, we introduce the following function

$$f^{(h)}(h \pm z) = \sum_{\nu=1}^{\infty} A_\nu \left(\frac{\partial H_\nu}{\partial y}\right)_{y=0} \cos\left[\beta_\nu (h \pm z)\right], \qquad (39)$$

in view of which the homogeneous portion of equation (33) may now be written as

*This is because numerical procedures tend to average local affects.

$$\frac{\partial^2}{\partial x^2}\{\, 2(1-\nu)\ f^{(h)}(h+z) + (h-z)\ \frac{\partial f^{(h)}(h+z)}{\partial z}\} + \frac{\partial^2}{\partial z^2}\{(h-z)\ \cdot$$

$$\frac{\partial f^{(h)}(h+z)}{\partial z}\} + \frac{\partial^2}{\partial x^2}\{\, 2(1-\nu)\ f^{(h)}(h-z) + (h+z)\ \frac{\partial f^{(h)}(h-z)}{\partial z}\}$$

$$+ \frac{\partial^2}{\partial z^2}\{\ (h+z)\ \frac{\partial f^{(h)}(h-z)}{\partial z}\} = 0\ ;\ \text{for all } -h \le z \le h\ . \qquad (40)$$

We seek next the solution of the above equation in the form:

$$f^{(h)}(h+z) = C_0(c-x)^{-1/2}\ (h-z)^{2\nu_0}\ \ F(\eta) \qquad (41)$$

where

$$\nu_0 = 1 - \nu \qquad (42)$$

and

$$\eta \equiv \frac{h-z}{\sqrt{(c-x)^2 + (h-z)^2}} \qquad (43)$$

Substituting, therefore, eq. (41) into (40) one finds that the equation is satisfied provided that the function $F(\eta)$ is a solution of the following ordinary differential equation:

$$(-\eta^7 + 2\eta^5 - \eta^3)\ F''' + (k\eta^6 + \ell\eta^4 + m\eta^2)\ F'' + (b\eta^5 + d\eta^3 + e\eta)\ F' + aF = 0 \qquad (44)$$

where

$$
\begin{aligned}
a &= 4\nu_0^2\,(1 - 2\,\nu_0) \\
b &= -\,6\,(1 + 3\,\nu_0) \\
d &= 12\nu_0^2 + 20\,\nu_0 + 21/4 \\
e &= -2\nu_0\,(1 + 6\,\nu_0) \\
k &= -\,6\,(1 + \nu_0) \\
l &= 4\,(2 + 3\,\nu_0) \\
m &= -\,2\,(1 + 3\nu_0)\ .
\end{aligned}
\qquad (45) - (51)
$$

Finally, the solution of equation (44) can be found by the application of the method of Frobenious, i.e.

$$F(\eta) = \sum_{n=0}^{\infty} C_n^{\star}\,\eta^{\beta+n}. \qquad (52)$$

Without going into the details, the indicial equation is found to be

$$\beta(\beta - 1)(\beta - 2) + (6\,\nu_0 + 2)\,\beta(\beta - 1) + 2\,\nu_0\,(1 + 6\,\nu_0)\,\beta$$

$$+ 4\,\nu_0^2\,(2\,\nu_0 - 1) = 0\ , \qquad (53)$$

the roots of which are:

$$\beta = \begin{cases} -2\,\nu_0 & \text{double root} \\ 1 - 2\,\nu_0 & \end{cases} \qquad (54)$$

One concluses, therefore, that*

$$f^{(h)}(h+z) = C_0 (c-x)^{-1/2} \{(c-x)^2 + (h-z)^2\}^{1-\nu} H(\eta) , \tag{55}$$

where $H(\eta)$ represents an even power series in η the behavior of which for small values of η, i.e. around the point $\eta = 0$, is shown in fig. 2. A similar asymptotic expansion around the points $\eta = \pm 1$ shows that**

$$H(\eta) = (1-\eta^2)^{\gamma} \sum_{n=0}^{\infty} \tilde{C}_n (1-\eta^2)^n \tag{56}$$

with $\gamma = \frac{3}{4}$. The behavior of H around the point $\eta = -1$, for example, is shown in fig. 3. It should be emphasized that there exist no singular points of any kind between zero and minus one. Hence the function $H(\eta)$, in between, represents a smooth curve. Pending a further analytical or numerical study of $H(\eta)$ the function has been approximated by the dotted curve.***

In view of eq. (55), the function of $f^{(h)}(h+z)$ has now been determined. While there are many ways that one can now use to complete the rest of the problem, the authors suggest the following method in order to extract an explicit analytical form of the solution. First, one needs to construct a harmonic function $\psi(x,y,h+z)$ such that

---

*The reader should note that this is only one out of the three solutions and that the double root leads to a logarithmic solution.

**The reader should note that the other two values of $\gamma = 0$ and $\gamma = \frac{1}{4}$ correspond to the logarithmic and the $\beta = 1 - 2\nu_0$ solutions, respectively.

***In the event that a numerical scheme is used, the matching of the two curves must be done with care.

$$\psi(x,y;h+z) = \sum_{\nu} A_{\nu} H_{\nu} \cos \beta_{\nu}(h+z) \tag{57}$$

and that the following boundary condition be satisfied

$$\left.\frac{\partial\psi}{\partial y}\right|_{\substack{y=0 \\ x<c}} = f^{(h)} + f^{(P)} , \tag{58}$$

where $f^{(P)}$ represents the particular solution of equation (40) and which can easily be obtained. This, however, is Newman's problem which in principle can be solved. To obtain, however, an explicit solution requires some effort. Be that as it may, the authors believe that this is feasible for there exists a considerable amount of knowledge related to the behavior of this type of a function.*

---

*For example, see eq. (36).

Discussion of the results

In order for us to see the form of the singularity which prevails at the vicinity of the corner point, we introduce the local spherical coordinates defined by the transformation:

$$
\begin{aligned}
c - x &= \rho \sin\theta \cos\phi \\
y &= \rho \sin\theta \sin\phi \\
h - z &= \rho \cos\theta ,
\end{aligned}
\tag{59}
$$

which lead to*

$$f^{(h)} (h + z) = C_0 \rho^{\frac{3}{2} - 2\nu} \frac{1}{\sqrt{\sin\theta}} H(\cos\theta) , \tag{60}$$

from which it is evident that the displacements are proportional to $\rho^{1/2 - 2\nu}$ and the stresses proportional to $\rho^{-1/2 - 2\nu}$. We observe, therefore, that this is the same result as that reported in reference [2].

Let us next examine the behavior of the function f in the inner layers of the plate, i.e. for all $x - c = \epsilon << h - z$

$$
\begin{aligned}
f^{(h)} (h + z) &= \frac{C_0}{\sqrt{\epsilon}} (h - z)^{2-2\nu} \left\{ 1 + \frac{\epsilon^2}{(h - z)^2} \right\}^{1-\nu} H\left(\frac{1}{\sqrt{1 + \frac{\epsilon^2}{(h - z)^2}}} \right) \\
&= \frac{C_0}{\sqrt{\epsilon}} (h - z)^{2-2\nu} H(1) + 0(\epsilon^{\circ}) ,
\end{aligned}
\tag{61}
$$

which again leads to the same type of results as those reported in ref. [2]. We now have, therefore, a better understanding of the type of expansions that equations (4) - (9) represent.

---

*The reader should recall that the function $f^{(h)}$ is defined on the plane $y = 0$.

It would be appropriate at this time to reexamine the question as to whether boundary conditions $\tau_{zz}^{(c)}$, $\tau_{xz}^{(c)}$ and $\tau_{yz}^{(c)}$ can indeed be satisfied at the plate faces.* An examination of these stresses shows that they can be expressed in terms of

$$(h - z) \frac{\partial^2 \psi}{\partial z^2} \quad \text{and} \quad \frac{\partial \psi}{\partial z}$$

and similar expressions.** The former is automatically satisfied by virtue of the factor $(h - z)$. The latter, after the odd derivative is taken, a factor $(h - z)$ is kicked out thus forcing it to vanish. The reader should recall that $f^{(h)}$ is an even function of $(h - z)$. All other terms, which are of a lesser order, are negated by the particular solution $[f^{(P)}]$. One concludes, therefore, that such an expansion can indeed satisfy the boundary conditions at the plate faces and, as a result, is admissible.

It would be interesting next to compute the displacement function $v(x,o,z)$ in the neighborhood of the corner point. Without going into the mathematical details one finds after some rearrangement that***

$$v^{(c)} = C_0 \sqrt{c - x} \quad \rho^{-2\nu} G(\eta) + \cdots \tag{62}$$

where

$$\begin{aligned} G(\eta) \equiv \; & [(4 \nu_0^2 - 4\nu_0) \eta^2 + (- 4 \nu_0^2 + \nu_0)] H + \\ & + [(3 - 4 \nu_0)\eta^3 + (4\nu_0 - \tfrac{3}{2})\eta] H' + (\eta^2 - 1) \eta^2 H''. \end{aligned} \tag{63}$$

---

*The reader should note that there was never a question as to whether the solution in the integral representation form did just that.

**The reader should also take into account eq. (30).

***The ... here indicate terms of higher order.

A plot of the function $G(\eta)$ is given in Fig. 3 for various Poisson's ratios. The value of $\eta = 0$ corresponds to the free surface while $\eta = -1$ corresponds to the crack front. The constant C_0 is proportional to the applied stress and a function of Poisson's ratio. Thus, although the amplitude of v may change, the trend will remain the same. An examination of the curve also shows that as one moves from the crack front to the free surfaces of the plate the crack widens, a result that meets our expectations.

Notice also that for zero Poisson's ratio the function G is constant and the displacement now becomes

$$\lim_{\nu \to 0} v^{(c)} (x,o,z) = \{ \lim_{\nu \to 0} C_0 \} \text{ const. } \sqrt{c - x} \ . \tag{64}$$

This result also meets our expectations for it represents an exact solution.

Let us next examine the behavior of the displacement in the neighborhood of the crack front, i.e. when $\eta = -1$. The function $H(\eta)$ then behaves like*

$$H(\eta) \sim (1 - \eta^2)^{3/4} \ ,$$

which then leads to

$$\begin{aligned} v^{(c)} &\sim \text{const. } \rho^{1/2-2\nu} \sqrt{\sin \phi} \ (1 - \eta^2)^{-1/4} + \cdots \\ &= \text{const. } \rho^{1/2-2\nu} . 1 + \cdots \end{aligned} \tag{65}$$

*See eq. (56).

Conclusions

As a result of this study, the authors draw the following conclusions.

First of all the displacements in the neighborhood of the corner point are at least proportional to

$$u_i \sim \rho^{1/2-2\nu} f_i(\theta,\phi) \tag{66}$$

and that the functions $f_i(\theta,\phi)$ do not vanish.

Second, while it is true that for Poisson's ratios greater* than 1/4 the displacements are infinite, the local strain energy however remains finite. Consequently, such a solution is admissible and, if it represents 'a solution', it is the only solution[4]. Infinite displacements are not new to the field of linear elasticity. For example, Cerruti's problem leads to higher order singularities (displacements $\sim 1/\rho$). It should be emphasized that the problem which the authors have solved is that of the complementary 3-D Griffith crack, where the crack faces now are subjected to a uniform internal pressure σ_0.

Third, the solution is not separable in spherical coordinates. Thus solutions that do invoke such an assumption may or may not lead to the correct or complete solution of the problem.

Fourth, physically what the solution really shows is that linear theory is inadequate in predicting the actual behavior of the material at such corner points. This, however, should not be interpreted as if no practical information can be extracted from such a solution. An example of this is the 2-D solution which predicts infinite stresses at the base of the crack.

Fifth, the material cannot sustain such large stresses at the corner and, as a result, it must relax. This relaxation is accomplished in two

*For ν = 1/4, the displacements are $\sim \rho^{1/2-2(1/4)} = \rho^0$. In this case $u \sim \ln\rho$.

ways. Initially the material yields, thus forming the shear lips, and simultaneously it fractures, thus forming the well known 'tunneling effect'. The authors believe that this fracture initiates first at the corner, where the material tries to smooth out the 90° angle and subsequently its crack advances in the manner shown on fig. 4.

Sixth, present experimental results may not be used to prove or disprove a solution for the following reasons. In experimental work a crack is usually introduced by mechanical means and is subsequently fatigued in order to attain a sharp crack tip. The process of fatigue, however, will undoubtedly smooth out these sharp corners thus altering the geometry of the problem. Moreover, if the displacements are indeed very large, the material will yield and the shear lips will be formed. In such a case, the principle of superposition is no longer valid and any such comparison may be meaningless, unless the load in the experimental procedure is applied to the crack faces in which case one then compares 'complementary' problems.

Seventh, it has been observed experimentally that on the free surfaces of a plate and around the neighborhood of the crack tip, a small dip appears during the process of deformation. This dip substantiates the fact that, whenever a load is applied to the plate, it forces the material to smooth out the ninety degree corner. Moreover, the larger the load is the more noticeable the dip is. As in the 2-D case, large forces of atomic or molecular attraction of the order of the "theoretical strength" prevail at such corners. These "cohesive forces", attempting to smooth out the right angle, pull the material towards the center of the plate thus forcing the crack front to 'buckle' and tunnel its way into the center of the plate.

Eighth, the authors believe that there exist two different types of solution expansions that intermix to form the various singularities of the problem. This phenomenon, coupled with the fact that the solution is not separable in spherical coordinates may offer an explanation as to why other authors obtained, numerically, results of a lesser order singularity.

Engineering fracture mechanics can deliver the methodology to compensate the inadequacies of conventional design concepts that were based on tensile strength or yield strength. While such criteria are adequate for many engineering structures, they are insufficient when the likelihood of cracks exists. Now, after approximately three decades of development, fracture mechanics has become a useful tool in design, particularly with high strength materials. Interestingly enough, this advancement has been based primarily on the 2-D solution of the Griffith crack problem. We believe that the 3-D solution of the Griffith crack problem will not only enable us to understand the mechanism of fracture propagation better, but it will also expand our horizons for future research and, hopefully, will contribute to the advancement of the field of fracture mechanics to higher levels of safe design against fracture. It is for these reasons that this study was undertaken.

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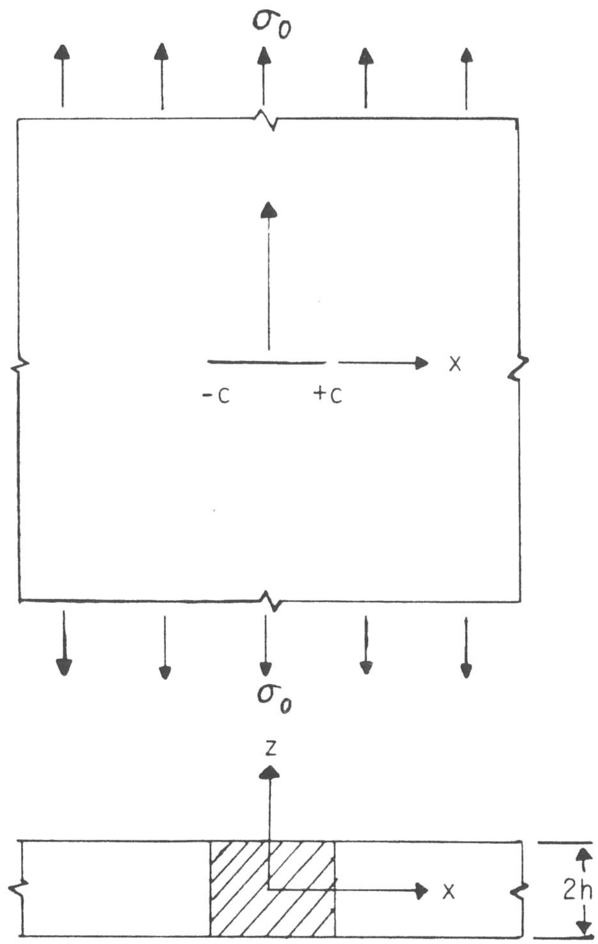

Fig. 1. Geometrical representation of an infinite cracked plate with thickness 2 h and length 2 c.

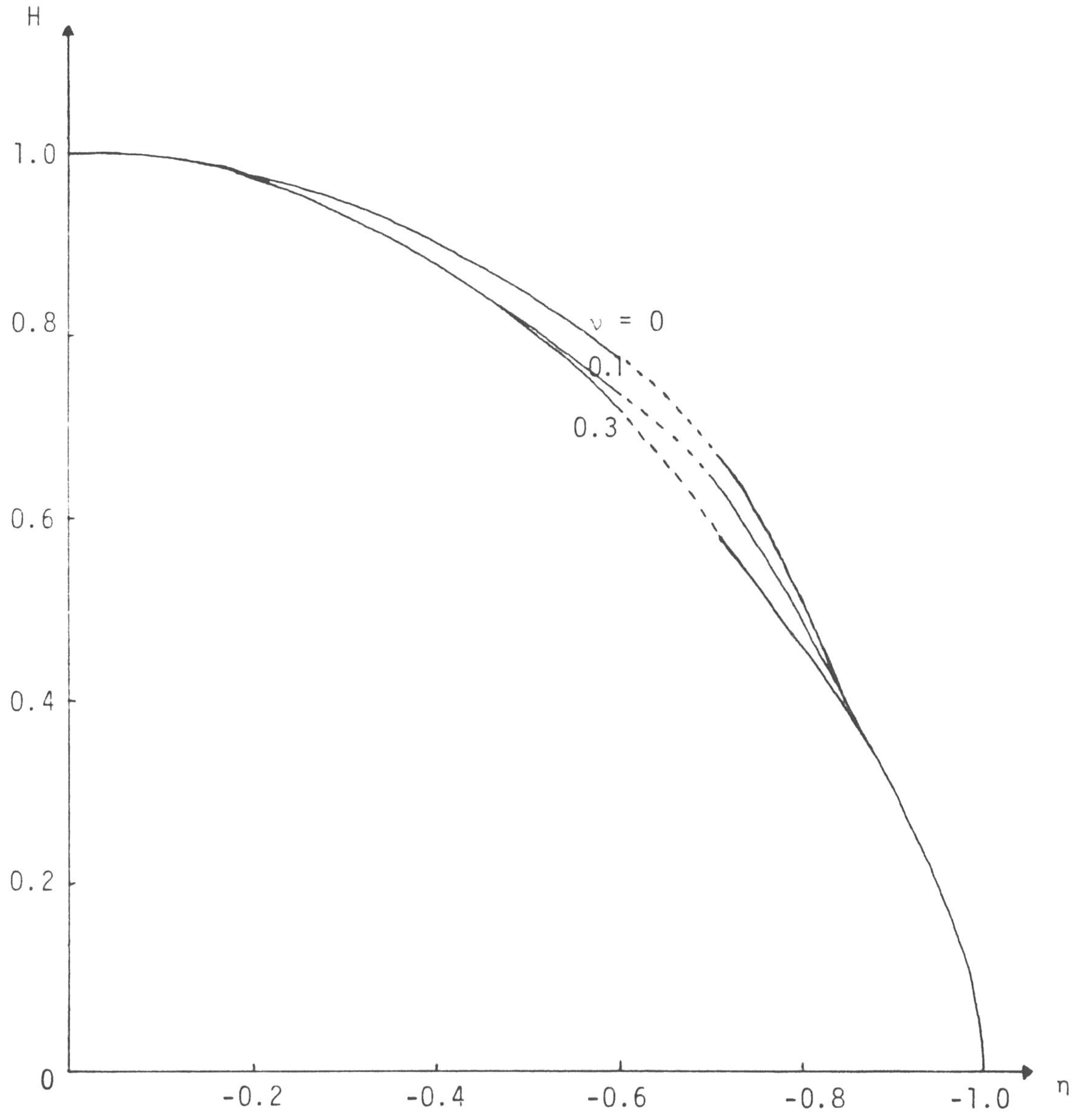

Fig. 2. Variation of H as a function of η.

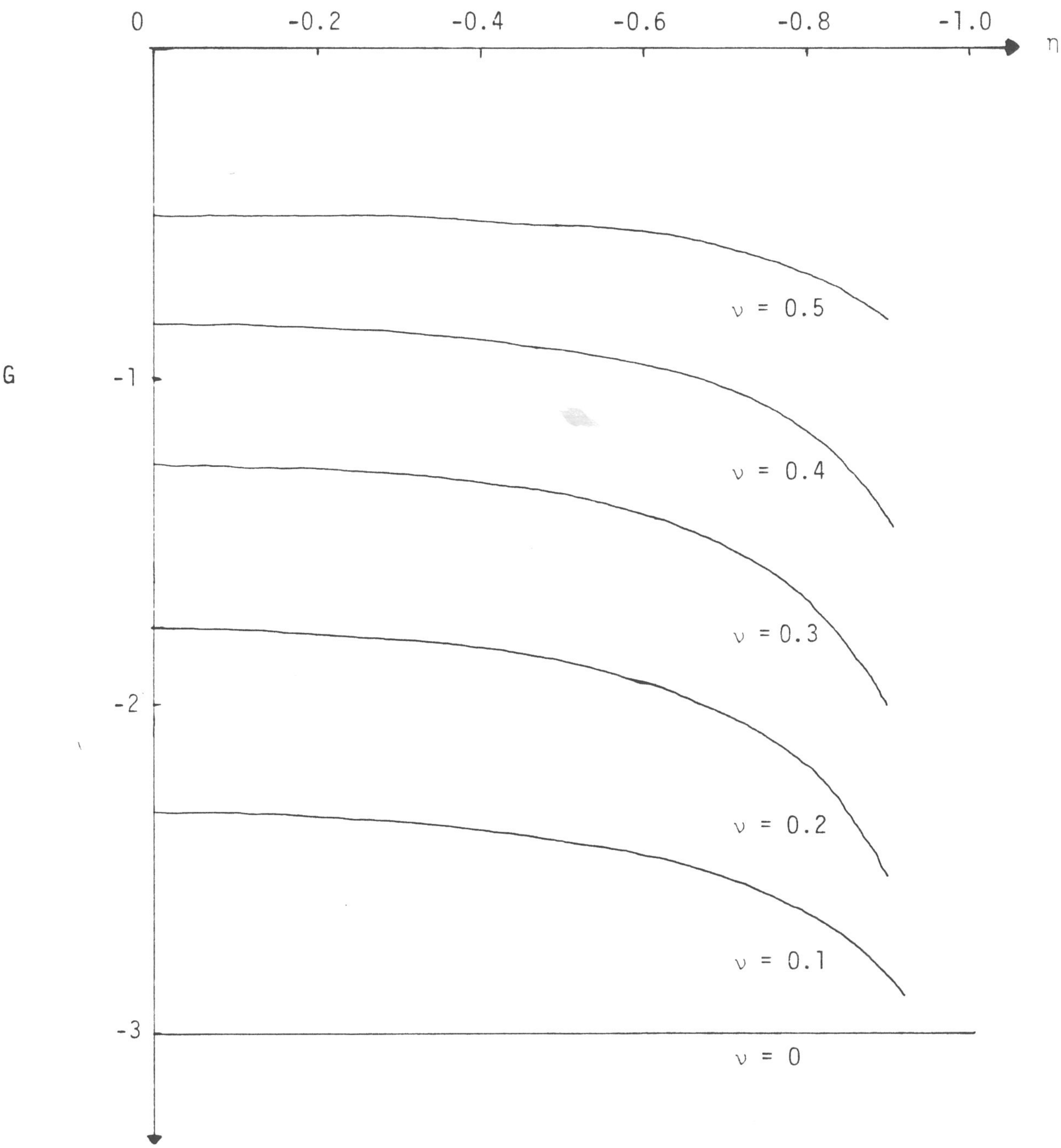

Fig. 3. Variation of G as a function of η.

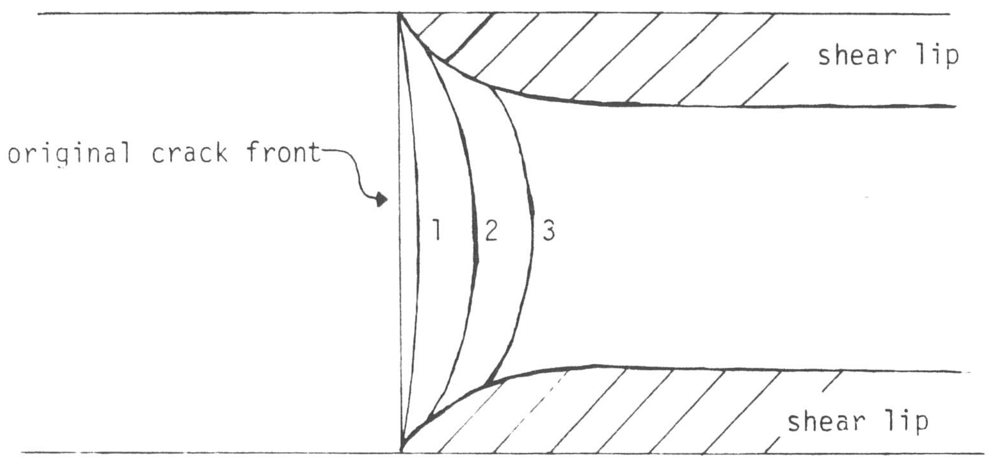

Fig. 4. Schematic diagram of crack propagation.

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This report discusses some further developments of an analytical solution to the 3-D Griffith crack problem. The analysis shows the stresses at the corner points to be singular of the order $(1/2 + 2\nu)$. Moreover, the stress boundary conditions at the plate faces are shown to be proportional to $(h-z)$, at the upper face, and to $(h+z)$, at the lower face.