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Method of solution of a class of three-dimensional elastostatic problems under mode I loading

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ABSTRACT
Using a Fourier integral transform, the problem of a cracked plate, of an arbitrary thickness $2h$ and subjected to a uniform external load of mode I, is reduced to that of the solution of a two-dimensional singular integral equation.

1. Introduction

In the field of fracture mechanics not very much theoretical work has been done in order to assess analytically the three-dimensional stress character which prevails at the base of a stationary crack. As a result, most of our current design criteria are based on already existing two-dimensional solutions and therefore are in general inadequate. For example, the common experimental observation of a change from ductile failure at the edge to brittle fracture at the center of a broken sheet material has so far defied analysis. Yet an orderly theoretical attack on the problem can provide important guidance to this and other phases of fracture research.

The mathematical difficulties, however, posed by three-dimensional fracture problems are substantially greater than those associated with plane stress or plane strain. Be that as it may, the author would like to investigate the subject further at least within the theory of linear elasticity. While he recognizes the fact that this theory cannot include the nonelastic behavior of the material at the crack tip per se, it can evince many characteristics of the actual behavior of a cracked plate, including those due to thickness. Thus the theory of elasticity is a logical fountainhead for detailed theoretical study.

2. Historical development

There exist in the literature very few analytical papers that deal specifically with the three-dimensional stress character at the base of a stationary crack. Moreover, in their present form these papers are not only incomplete but also contradictory. As a result, much controversy and many doubts have been raised. It is appropriate, therefore, to discuss these papers and their respective results in chronological order.

In 1972, Benthem, using the method of separation of variables*, was able to solve for the stress distribution in the neighborhood of the corner point** of a quarter plane

* This method was fully articulated by M.L. Williams [1] for classical planar elasticity in order to establish the singular behavior at re-entrant corners.

** That is the point where the crack front meets the free surface of the half space.

crack. His results [2] show that the stresses there behave like $\rho^{-\alpha}$, where $0.500 \leq \alpha \leq 0.709$. In order to obtain the order of the singularity, Benthem had to truncate an infinite system which, in turn, he solved for the eigenvalues numerically. This approach, however, raises three important questions: One, is the solution really separable, particularly in θ and ϕ? Two, is the solution thus obtained complete? Three, should the numerical determination of the singularity from a truncated system be trusted? Unfortunately, Benthem has provided no answers to any of the above important and difficult questions.

A few years later, Folias, using a method developed by Lur'e [3] and the application of Fourier Integral Transforms, was able to solve [4] Navier's equations for a more complicated problem, that of the 3-D Griffith crack (see Fig. 1). The integrals were subsequently expanded asymptotically and the stress field, valid in the very inner layers* of the plate, was recovered. From the results, one concludes that in the very inner layers of the plate:

(1) the stresses possess the usual singularity,
(2) the stresses possess the usual angular distribution,
(3) the stress intensity factor K_{I} is a function of z,
(4) exact plane strain conditions exist only on the plane $z = 0$,
(5) a pseudo plane strain state exists and the equation

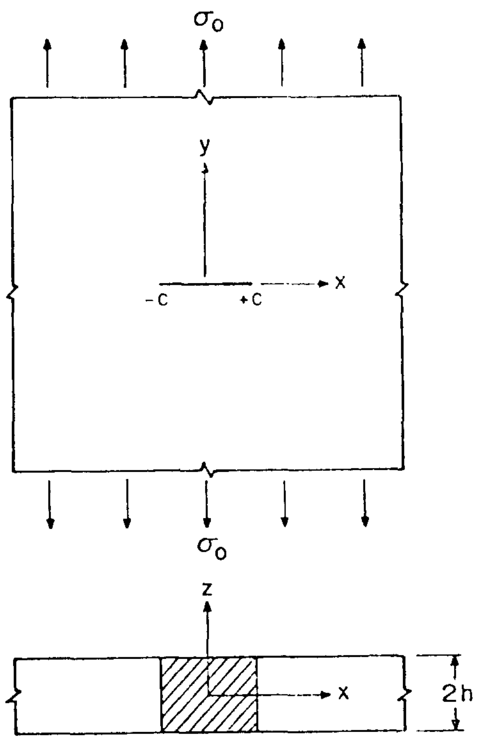

Figure 1. Geometrical representation of an infinite cracked plate with thickness $2h$ and crack length $2c$.

* The reader should note that the asymptotic expansions are only valid for $(z/h) \ll 1$ and for $c/h \ll 1$. This is because h was assumed to be very large so that a perturbation about the well-known plane-strain solution could be made.

$$\sigma_z = \nu(\sigma_x + \sigma_y)$$

is satisfied,

(6) as the plate thickness $2h \to \infty$, the plane solution is recovered,

(7) as Poisson's ratio $\nu \to 0$, the plane stress solution is recovered.

Furthermore, he was able to show that at the corner the stresses are proportional to

$$\rho^{-(1/2+2\nu)} f_{ij}(\theta, \phi).$$

In order to recover the value of the singularity, Folias solved analytically a difference-differential equation. Unfortunately, because of the enormous difficulties which the integral representations presented at the corner, he was unable at the time to recover the functions $f_{ij}(\theta, \phi)$ explicitly.

It should be emphasized that Folias's main result at the corner should be interpreted as "the singularity at the corner can at most be of the order $(\frac{1}{2} + 2\nu)$". This is because the functions $f_{ij}(\theta, \phi)$ could very well be of the type that do vanish* in the neighborhood of the corner point. Thus Folias's result may or may not be in contradiction with Benthem's.

Researchers in the field of fracture mechanics, however, were unwilling to accept the possibility of an infinite displacement field on the basis of physical intuition. Consequently, the results were considered highly controversial and the following two legitimate questions were raised**: Is the solution really complete? Two, do the series representations converge? Unfortunately, Folias provided no answers to any of the above questions.

In 1976, Kawai [7], using the method of separation of variables was able to obtain an alternate solution to Benthem's problem. Although the method of approach is essentially the same as that of Benthem's, his results are definitely contradictory***. His results show that at the corner the stresses behave like $\rho^{-\alpha}$, where $\frac{1}{2} \leq \alpha < 1$. In determining the singularity, Kawai used the collocation method in order to satisfy the three boundary condition on the free surface. Thus, as in Benthem's case, the same questions apply to this work also.

A few months later, Benthem discovered that his previously reported solution was incomplete and that his new results [8] now read

$$\sigma_{ij} \sim \rho^{-\alpha} \text{ with } 0 \leq \alpha \leq \tfrac{1}{2}.$$

Here again, the same questions raised during his previous work apply too.

Finally, in 1977 Kawai [9] reported an error in his previous analysis and although the correction affected slightly the value of α the trend essentially remained the same.

In the meantime, Folias also discovered that his solution of the difference-differential equation was not quite complete either****. The correction, however, does not directly alter the basic result at the corner.

It is interesting to note that Kawai does recover the same singularity that Folias reported. The singularity $(-\frac{1}{2} - 2\nu)$, however, disappears as he considers more and more terms in his collocation scheme but at the same time he experiences con-

* The reader should note that this result was actually obtained by 'marching out' the solution from the inner to the outer layers, and as a result such a hypothesis may not be totally unreasonable. See also comments on p. 5.

** See Discussion of paper by Benthem and Koiter [5] and author's Closure [6].

*** Mathematically, Kawai's method of construction of the solution is more systematic than that of Benthem's.

**** This is not to be confused with the question of completeness of the solution to Navier's equations, i.e. Eqns. (52)–(54) Ref. [4]. The corrected result to (85) of [4] is given in Appendix I.

vergence problems. This observation strengthens, perhaps, the hypothesis that Folias's $f_{ij}(\theta, \phi)$ functions do indeed vanish in the neighborhood of the corner point and that most likely are needed in the very inner layers of the plate. The latter has also been observed by Newman [10] for (c/h) ratios less than one, which is comparable with the asymptotic expansion used by Folias.

Be that as it may, the presence of a third solution obscured the issue even further and essentially raised more questions than gave answers. So the controversy still remains.

3. Purpose of present work

In view of the preceding, it is evident that mathematical rigour becomes essential if one is to avoid any possible pitfalls. As a result, the author decided to seek the answers to the following two important questions first:

(i) Is the solution of this notoriously difficult problem unique? And if so, under what conditions?
(ii) Is the solution to Navier's equations as given by the author in [4], i.e. Eqns. (52)–(54), general enough to represent the solution of this practical problem?

The answers to both of the above questions were given by Prof. Calvin Wilcox.

First of all, he was successful in proving [11] that a displacement field that satisfies the condition of local finite energy is unique. This of course is quite a departure from our traditional 2-D fracture mechanics thinking, for the displacements may now be allowed to be singular. Consequently, one may not a priori assume them to be finite as it is customarily done. In general, such an assumption makes the class of solutions too restrictive and, as a result, one may not find a solution to the problem. On the other hand, the solution could very well give finite displacements everywhere! Be that as it may, physical intuition should be used with extreme caution.

Second, he was able to show [12] that the Fourier integral expressions* representing the general solution to Navier's equations are complete and, furthermore, the 'symbolic method' used is justifiable. In order to prove this, he used a double Fourier integral transform in x and y and subsequently a contour integration to recover precisely the same expressions as those reported by Folias in [4].

Finally, it remains to determine explicitly the stress field ahead of the crack tip and throughout the thickness of the plate. In [4], the author, by the use of analytic continuation, attempted to 'march out' the solution from the inner to the outer layers of the plate. Although in principle this seems feasible, in practice it is very difficult and most of all tedious. Moreover, questions of convergence will inevitably be raised. As a result, in this paper we will use an alternate and more elegant approach in order to complete the problem.

By finding the biorthogonal relation for the eigenvectors, we will set up a double integral equation for the unknown function v, which, physically, represents the crack opening displacement. The advantages of this new approach over that of [4] are:

(i) we are now seeking the solution to one equation only,
(ii) the unknown function is real and furthermore has physical meaning,
(iii) the kernel of the integral equation is independent of the shape of the crack**.

* See Eqns. (52)–(54) of [4].
** In this analysis we restrict ourselves to planar and symmetric cracks subjected to mode I loadings.

4. Formulation of the problem

Consider the equilibrium of a homogeneous, isotropic, elastic plate which occupies the space $|x| < \infty$, $|y| < \infty$, $|z| < h$ and contains a plane crack in the x-z- plane (see Fig. 1). The crack faces, defined by $|x| < c$, $y = 0^{\pm}$, $z \leq h$, and the plate faces $|z| = h$ are free of stress and constraint. Loading is applied on the periphery of the plate $|x|$, $|y| \to \infty$ and is given by

$$\sigma_x = \tau_{xy} = \tau_{yz} = 0, \quad \sigma_y = \bar{\sigma}_0.$$

In the absence of body forces, the coupled differential equations governing the displacement functions u, v, and w are

$$\frac{m}{m-2}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)e + \nabla^2(u, v, w) = 0 \tag{1)–(3}$$

where ∇^2 is the Laplacian operator, $m \equiv 1/\nu$, ν is Poisson's ratio,

$$e \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{4}$$

and the stress-displacement relations are given by Hooke's law as:

$$\sigma_x = 2G\left\{\frac{\partial u}{\partial x} + \frac{e}{m-2}\right\}, \ldots, \tau_{xy} = G\left\{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right\}, \ldots \tag{5)–(10}$$

with G being the shear modulus.

As to boundary conditions, one must require that at:

$$|x| < c,\ y = 0^{\pm},\ |z| \leq h: \quad \tau_{xy} = \tau_{yz} = \sigma_y = 0 \tag{11}$$

$$|z| = h \quad : \quad \tau_{xz} = \tau_{yz} = \sigma_z = 0 \tag{12}$$

$$|y| \to \infty \text{ and all } x \quad : \quad \tau_{xy} = \tau_{yz} = 0,\ \sigma_y = \bar{\sigma}_0 \tag{13}$$

$$|x| \to \infty \quad : \quad \sigma_x = \tau_{xy} = \tau_{zx} = 0. \tag{14}$$

It is found convenient to seek the solution to the crack plate problem in the form

$$u = u^{(P)} + u^{(C)} \text{ etc.,} \tag{15}$$

where the first component represents the usual "undisturbed" or "particular" solution of a plate without the presence of a crack. Such a particular solution can easily be constructed and for the particular problem at hand is

$$\begin{aligned} u^{(P)} &= -\frac{\bar{\sigma}_0}{2G\Delta}(m-2)^2 x, \\ v^{(P)} &= -[1-(m-1)^2]\frac{\bar{\sigma}_0(m-2)}{2G\Delta}\, y \\ w^{(P)} &= -(m-2)^2\frac{\bar{\sigma}_0}{2G\Delta}\, z \end{aligned} \tag{16}$$

where

$$\Delta = (m-1)^3 - 3(m-1) + 2.$$

5. Method of solution

The complementary solution to Navier's equations, subject to the corresponding boundary conditions (12) and (13), is given by [4] as:

(i) *Complementary displacements**

$$u^{(c)} = \int_0^\infty \Bigg\{\left(p_1 + |y|Q_1 + \frac{1}{m+1} z^2 s Q_1\right) e^{-s|y|}$$
$$-\frac{1}{m-2}\sum_{\nu=1}^\infty \Gamma_\nu \frac{e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{\sqrt{s^2+\beta_\nu^2}} \cos(\beta_\nu h)[(m-2+m\cos^2(\beta_\nu h))\cos(\beta_\nu z) -$$
$$- m\beta_\nu z \sin(\beta_\nu z)] + \sum_{n=1}^\infty S_n e^{-\sqrt{s^2+\alpha_n^2}|y|}\cos(\alpha_n z)\Bigg\} \sin(xs)\,ds \tag{17}$$

$$v^{(c)} = \mp \int_0^\infty \Bigg\{\left(-\frac{3m-1}{m+1}\frac{Q_1}{s} - P_1 - |y|Q_1 - \frac{1}{m+1} s z^2 Q_1\right) e^{-s|y|}$$
$$+\frac{1}{m-2}\sum_{\nu=1}^\infty \Gamma_\nu \frac{e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{s} \cos(\beta_\nu h)[(m-2+m\cos^2(\beta_\nu h))\cos(\beta_\nu z) -$$
$$- m\beta_\nu z\sin(\beta_\nu z)] - \sum_{n=1}^\infty S_n \frac{s}{\sqrt{s^2+\alpha_n^2}} e^{-\sqrt{s^2+\alpha_n^2}|y|}\cos(\alpha_n z)\Bigg\}\cos(xs)\,ds. \tag{18}$$

$$w^{(c)} = \int_0^\infty \Bigg\{\left(\frac{2}{m+1} z Q_1\right) e^{-s|y|} + \frac{1}{m-2}\sum_{\nu=1}^\infty \beta_\nu \Gamma_\nu \frac{e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{s\sqrt{s^2+\beta_\nu^2}}\cos(\beta_\nu h)$$
$$\cdot [(2m-2-m\cos^2(\beta_\nu h))\sin(\beta_\nu z) - m\beta_\nu z\cos(\beta_\nu z)]\Bigg\}\cos(xs)\,ds. \tag{19}$$

with corresponding stress:

(ii) *complementary stresses*

$$\frac{\sigma_z^{(c)}}{2G} = \frac{m}{m-2}\int_0^\infty \sum_{\nu=1}^\infty \Gamma_\nu \beta_\nu^2 \frac{e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{s\sqrt{s^2+\beta_\nu^2}}\cos(\beta_\nu h)$$
$$\times [\sin^2(\beta_\nu h)\cos(\beta_\nu z) + \beta_\nu z\sin(\beta_\nu z)]\cos(xs)\,ds \tag{20}$$

$$\frac{\tau_{xz}^{(c)}}{G} = \int_0^\infty \Bigg\{\frac{2m}{m-2}\sum_{\nu=1}^\infty \beta_\nu\Gamma_\nu \frac{e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{\sqrt{s^2+\beta_\nu^2}}\cos(\beta_\nu h)[\cos^2(\beta_\nu h)\sin(\beta_\nu z)$$
$$+ \beta_\nu z\cos(\beta_\nu z)] - \sum_{n=1}^\infty S_n\alpha_n e^{-\sqrt{s^2+\alpha_n^2}|y|}\sin(\alpha_n z)\Bigg\}\sin(xs)\,ds. \tag{21}$$

$$\frac{\tau_{yz}^{(c)}}{G} = \pm\int_0^\infty \Bigg\{\frac{2m}{m-2}\sum_{\nu=1}^\infty \beta_\nu\Gamma_\nu \frac{e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{s}\cos(\beta_\nu h)[\cos^2(\beta_\nu h)\sin(\beta_\nu z)$$
$$+ \beta_\nu z\cos(\beta_\nu z)] - \sum_{n=1}^\infty S_n \frac{\alpha_n s\, e^{-\sqrt{s^2+\alpha_n^2}|y|}}{\sqrt{s^2+\alpha_n^2}}\sin(\alpha_n z)\Bigg\}\cos(xs)\,ds. \tag{22}$$

$$\frac{\sigma_x^{(c)}}{2G} = \int_0^\infty \Bigg\{\left(sP_1 + |y|sQ_1 + \frac{1}{m+1}s^2z^2Q_1 - \frac{2}{m+1}Q_1\right)e^{-s|y|}$$
$$+\frac{2}{m-2}\sum_{\nu=1}^\infty \beta_\nu^2\Gamma_\nu \frac{e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{s\sqrt{s^2+\beta_\nu^2}}\cos(\beta_\nu h)\cos(\beta_\nu z)$$
$$-\frac{1}{m-2}\sum_{\nu=1}^\infty \Gamma_\nu \frac{s e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{\sqrt{s^2+\beta_\nu^2}}\cos(\beta_\nu h)$$
$$\times [(m-2+m\cos^2(\beta_\nu h)\cos(\beta_\nu z) - m\beta_\nu z\sin(\beta_\nu z)]$$
$$+\sum_{n=1}^\infty S_n\, s e^{-\sqrt{s^2+\alpha_n^2}|y|}\cos(\alpha_n z)\Bigg\}\cos(xs)\,ds. \tag{23}$$

* These complementary displacements represent a 'general enough', or 'complete', solution for the satisfaction of the remaining boundary conditions. For a discussion of this, see [12].

$$\begin{aligned}\frac{\sigma_y^{(c)}}{2G} = \int_0^\infty \Bigg\{ &\left(-\frac{2m}{m+1}Q_1 - sP_1 - |y|sQ_1 - \frac{1}{m+1}s^2z^2Q_1\right)e^{-s|y|} \\ &+ \frac{2}{m-2}\sum_{\nu=1}^{\infty}\beta_\nu^2\Gamma_\nu \frac{e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{s\sqrt{s^2+\beta_\nu^2}}\cos(\beta_\nu h)\cos(\beta_\nu z) \\ &+ \frac{1}{m-2}\sum_{\nu=1}^{\infty}\Gamma_\nu \frac{\sqrt{s^2+\beta_\nu^2}}{s}e^{-\sqrt{s^2+\beta_\nu^2}|y|}\cos(\beta_\nu h) \\ &\times[(m-2+m\cos^2(\beta_\nu h))\cos(\beta_\nu z) - m\beta_\nu z\sin(\beta_\nu z)] \\ &- \sum_{n=1}^{\infty} S_n s\, e^{-\sqrt{s^2+\alpha_n^2}|y|}\cos(\alpha_n z)\Bigg\}\cos(xs)\,ds. \end{aligned} \tag{24}$$

$$\begin{aligned}\frac{\tau_{xy}^{(c)}}{G} = \mp\int_0^\infty \Bigg\{ &\left(2\left(\frac{m-1}{m+1}\right)Q_1 + 2sP_1 + \frac{2}{m+1}z^2s^2Q_1 + 2|y|sQ_1\right)e^{-s|y|} \\ &- \frac{2}{m-2}\sum_{\nu=1}^{\infty}\Gamma_\nu e^{-\sqrt{s^2+\beta_\nu^2}|y|}\cos(\beta_\nu h) \\ &\times[(m-2+m\cos^2(\beta_\nu h))\cos(\beta_\nu z) - m\beta_\nu z\sin(\beta_\nu z)] \\ &+ \sum_{n=1}^{\infty} S_n \frac{2s^2+\alpha_n^2}{\sqrt{s^2+\alpha_n^2}}e^{-\sqrt{s^2+\alpha_n^2}|y|}\cos(\alpha_n z)\Bigg\}\sin(xs)\,ds. \end{aligned} \tag{25}$$

where the $\pm$ signs refer to $y>0$ and $y<0$ respectively and the constants P_1, Q_1, Γ_ν and S_n are to be determined from the remaining boundary conditions. Moreover, $\alpha_n = n\pi/h$ $(n = 1, 2, 3\ldots)$, and β_ν are the roots of the equation

$$\sin(2\beta_\nu h) = -(2\beta_\nu h): \tag{26}$$

This equation has an infinite number of complex roots which appear in groups of four, one in each quadrant of the complex plane and only two of each group of four roots are relevant to the present work. These are chosen to be the complex conjugate pairs with positive real parts. The only real root $\beta_\nu = 0$ must be ignored*.

By direct substitution, it can easily be ascertained that the above complementary displacements satisfy Navier's equations and furthermore the corresponding stresses $\sigma_z^{(c)}$, $\tau_{zx}^{(c)}$, $\tau_{yz}^{(c)}$ do vanish at the plate faces $z = \pm h$.

Finally, if we consider the following two combinations to vanish**

$$\frac{2m}{m-2}\sum_{\nu=1}^{\infty}\frac{\Gamma_\nu}{s}\cos(\beta_\nu h)[\sin^2(\beta_\nu h)\cos(\beta_\nu z) + \beta_\nu z\sin(\beta_\nu z)] + \sum_{n=1}^{\infty}\frac{s\,S_n}{\sqrt{s^2+\alpha_n^2}}\cos(\alpha_n z) + \frac{4m}{m-2}\sum_{\nu=1}^{\infty}\frac{\Gamma_\nu}{s} = 0. \tag{27}$$

and

$$\begin{aligned}&\frac{2}{m-2}\sum_{\nu=1}^{\infty}\Gamma_\nu\cos(\beta_\nu h)[(m-2+m\cos^2(\beta_\nu h))\cos(\beta_\nu z) - m\beta_\nu z\sin(\beta_\nu z)] \\ &\quad - \sum_{n-1}^{\infty}\frac{2s^2+\alpha_n^2}{\sqrt{s^2+\alpha_n^2}}S_n\cos(\alpha_n z) - \frac{2}{1+m}s^2z^2Q_1 - 2sP_1 - 2\frac{m-1}{m+1}Q_1 = 0 \end{aligned} \tag{28}$$

for all $|z| \le h$, then two of the remaining stress boundary conditions are satisfied automatically, i.e.

$$\tau_{xy}^{(c)} = \tau_{yz}^{(c)} = 0 \quad \text{for all} \quad x, |z| \le h \quad \text{and} \quad y = 0.$$

We will suppress for the time being the satisfaction of the last boundary condition and will focus our attention to the continuity conditions.

* The first few roots are tabulated in Appendix II.

** Notice that the derivative of (27) with respect to z leads to the integrand of (22).

As it can easily be seen, all continuity conditions are satisfied if one considers the following two combinations to vanish

$$\int_0^\infty \left\{-\frac{3m-1}{m+1}\frac{Q_1}{s} - P_1 - \frac{1}{m+1} sz^2 Q_1 + \frac{1}{m-2}\sum_{\nu=1}^{\infty}\frac{\Gamma_\nu}{s}\cos(\beta_\nu h) \right.$$
$$\times [(2m-2+m\sin^2(\beta_\nu h))\cos(\beta_\nu z)$$
$$\left. + m\beta_\nu z\sin(\beta_\nu z)] + \frac{4m}{m-2}\sum_{\nu=1}^{\infty}\frac{\Gamma_\nu}{s}\right\}\cos(xs)\,\mathrm{d}s = 0; \quad |x|>c, \forall |z|<h. \tag{29}$$

and

$$\int_0^\infty \left\{-\frac{4m}{m+1}Q_1 + \frac{2m}{m-2}\sum_{\nu=1}^{\infty}\frac{\Gamma_\nu}{s^2}\beta_\nu^2[(1+\cos^2(\beta_\nu h))\cos(\beta_\nu z) - \beta_\nu z\sin(\beta_\nu z)]\right\}$$
$$\times \sin(xs)\,\mathrm{d}s = 0; \quad |x|>c, \forall|z|<h. \tag{30}$$

which by Fourier inversion lead to:

$$-\frac{3m-1}{m+1}\frac{Q_1}{s} - P_1 - \frac{1}{m+1}sz^2Q_1 + \frac{1}{m-2}\sum_{\nu=1}^{\infty}\frac{\Gamma_\nu}{s}\cos(\beta_\nu h)$$
$$\cdot [(2m-2+m\sin^2(\beta_\nu h))\cos(\beta_\nu z) + m\beta_\nu z\sin(\beta_\nu z)]$$
$$+\frac{4m}{m-2}\sum_{\nu=1}^{\infty}\frac{\Gamma_\nu}{s} = \mp\frac{2}{\pi}\int_0^c v(\xi,0,z)\cos(s\xi)\,\mathrm{d}\xi \tag{31}$$

and*

$$-\frac{4m}{m+1}Q_1 + \frac{2m}{m-2}\sum_{\nu=1}^{\infty}\frac{\Gamma_\nu}{s^2}\beta_\nu^2\cos(\beta_\nu h)[(1+\cos^2(\beta_\nu h))\cos(\beta_\nu z) - \beta_\nu z\sin(\beta_\nu z)]$$
$$= \mp\frac{2}{\pi}\int_0^c\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial \xi}\right)_{y=0}\sin(s\xi)\,\mathrm{d}\xi$$
$$= \pm\frac{2}{\pi}\int_0^c 2\left(\frac{\partial v}{\partial \xi}\right)_{y=0}\sin(s\xi)d\xi = \mp\frac{4s}{\pi}\int_0^c v(\xi,0,z)\cos(s\xi)\,\mathrm{d}\xi. \tag{32}$$

Adopting next the following definitions

$$Z_\nu^{(1)}(z) \equiv (\beta_\nu h)^2[\beta_\nu h\sin(\beta_\nu h)\cos(\beta_\nu z) - \beta_\nu z\cos(\beta_\nu h)\sin(\beta_\nu z)] \tag{33}$$

$$Z_\nu^{(3)}(z) \equiv -(\beta_\nu h)^2[\beta_\nu h\sin(\beta_\nu h)\cos(\beta_\nu z) - \beta_\nu z\cos(\beta_\nu h)\sin(\beta_\nu z)]$$
$$-2(\beta_\nu h)^2\cos(\beta_\nu h)\cos(\beta_\nu z) \tag{34}$$

$$f^{(1)}(z) \equiv \mp\frac{2h^2}{\pi s}\left(\frac{m-2}{m-1}\right)\int_0^c\left\{\frac{\partial^2 v}{\partial z^2} - \frac{1}{m}s^2 v\right\}\cos(s\xi)\,\mathrm{d}\xi \tag{35}$$

$$f^{(3)}(z) \equiv \pm\frac{2sh^2}{\pi}\left(\frac{m-2}{m}\right)\int_0^c v\cos(s\xi)\,\mathrm{d}\xi - 2\left(\frac{m-2}{m+1}\right)Q_1h^2, \tag{36}$$

Eqn. (32) and the second derivative of (31) with respect to z become

$$-\frac{m}{m-2}\sum_{\nu=1}^{\infty}\frac{\Gamma_\nu}{s^2}Z_\nu^{(3)}(z) = \mp\frac{2sh^2}{\pi}\int_0^c v(\xi,0,z)\cos(s\xi)\,\mathrm{d}\xi + \frac{2m}{m+1}Q_1h^2 \tag{37}$$

and**

$$-\left(\frac{m-1}{m-2}\right)\sum_{\nu=1}^{\infty}\frac{\Gamma_\nu}{s^2}\{Z_\nu^{(1)}(z) + Z_\nu^{(3)}(z)\}$$
$$= \pm\frac{2h^2}{\pi s}\int_0^c\left\{\frac{\partial^2 v}{\partial z^2} - s^2 v\right\}_{y=0}\cos(s\xi)\,\mathrm{d}\xi + 2\left(\frac{m-1}{m+1}\right)Q_1h^2 \tag{38}$$

* The reader should note that Eqns. (31) and (32) automatically satisfy (28).

** Notice that the continuity conditions are to be satisfied in the interior of the plate only.

respectively, which upon simplifying one has

$$\sum_{\nu=1}^{\infty}\left(\frac{\Gamma_\nu}{s^2}\right)\begin{bmatrix}Z_\nu^{(3)}(z)\\ Z_\nu^{(1)}(z)\end{bmatrix}=\begin{bmatrix}f^{(3)}(z)\\ f^{(1)}(z)\end{bmatrix} \tag{39}$$

Next, following [13], we can construct the biorthognal relations

$$W_\nu^{(4)}(z)\equiv -\beta_\nu^* z\cos(\beta_\nu^* h)\sin(\beta_\nu^* z)+\beta_\nu^* h\sin(\beta_\nu^* h)\cos(\beta_\nu^* z) \tag{40}$$

and

$$W_\nu^{(2)}(z)\equiv -\beta_\nu^* z\cos(\beta_\nu^* h)\sin(\beta_\nu^* z)+[\beta_\nu^* h\sin(\beta_\nu^* h)-2\cos(\beta_\nu^* h)]\cos(\beta_\nu^* z), \tag{41}$$

where β_ν^* stands for the complex conjugate of the β_ν roots. The orthogonality condition now reads

$$\sum_{\nu=1}^{\infty}\left(\frac{\Gamma_\nu}{s^2}\right)\frac{1}{h}\int_{-h}^{h}[W_k^{*(4)}(\eta)W_k^{*(2)}(\eta)]\begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}\begin{bmatrix}Z_\nu^{(1)}(\eta)\\ Z_\nu^{(3)}(\eta)\end{bmatrix}\mathrm{d}\eta$$

$$=\frac{1}{h}\int_{-h}^{h}[W_k^{*(4)}(\eta)W_k^{*(2)}(\eta)]\begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}\begin{bmatrix}f^{(1)}(\eta)\\ f^{(3)}(\eta)\end{bmatrix}\mathrm{d}\eta, \tag{42}$$

or

$$\left(\frac{\Gamma_\nu}{s^2}\right){}_eK_\nu=\frac{1}{h}\int_{-h}^{h}\{W_\nu^{*(4)}(\eta)f^{(1)}(\eta)+[2W_\nu^{*(4)}(\eta)-W_\nu^{*(2)}(\eta)]f^{(3)}(\eta)\}\,\mathrm{d}\eta \tag{43}$$

where for simplicity we have defined

$${}_eK_\nu\equiv\frac{1}{h}\int_{-h}^{h}\{W_\nu^{*(4)}(\eta)Z_\nu^{(1)}(\eta)+[2W_\nu^{*(4)}(\eta)-W_\nu^{*(2)}(\eta)]Z_\nu^{(3)}(\eta)\}\,\mathrm{d}\eta. \tag{44}$$

Finally, in view of (33)–(36), (40)–(41) and (43)–(44), one finds after some simple calculations that

$${}_eK_\nu=-4(\beta_\nu h)^2\cos^4(\beta_\nu h) \tag{45}$$

and

$$\begin{aligned}{}_eK_\nu\left(\frac{\Gamma_\nu}{s^2}\right)=\pm\frac{2}{\pi sh}\left(\frac{m-2}{m-1}\right)\int_0^c\int_{-h}^{h}v(\xi,0,\eta)\cos(s\xi)\,\{(hs)^2[\beta_\nu h\sin(\beta_\nu h)\cos(\beta_\nu\eta)\\ -\beta_\nu\eta\cos(\beta_\nu h)\sin(\beta_\nu\eta)+2\left(\frac{m-1}{m}\right)\cos(\beta_\nu h)\cos(\beta_\nu\eta)]\\ +(h\beta_\nu)^2[\beta_\nu h\sin(\beta_\nu h)\cos(\beta_\nu\eta)-\beta_\nu\eta\cos(\beta_\nu h)\sin(\beta_\nu\eta)\\ +2\cos(\beta_\nu h)\cos(\beta_\nu\eta)]\}\,\mathrm{d}\eta\,\mathrm{d}\xi.\end{aligned} \tag{46}$$

Similarly, from (27) and (32), we find that

$$\frac{S_n}{\sqrt{s^2+\alpha_n^2}}(\alpha_n h)^2=\mp\frac{4sh^2}{\pi h}\int_0^c\int_{-h}^{h}v(\xi,0,\eta)\cos(s\xi)\cos(\alpha_n\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi \tag{47a}$$

and

$$Q_1=\pm\frac{m+1}{2m}\frac{(sh)}{\pi}\int_0^c\int_{-h}^{h}v(\xi,0,\eta)\cos(s\xi)\,\mathrm{d}\eta\,\mathrm{d}\xi. \tag{47b}$$

Returning now to the last boundary condition, we require that*

* Where we have made use of (28).

$$\int_0^\infty \Big\{ -Q_1 + \frac{m}{m-2}\sum_{\nu=1}^{\infty} \frac{\Gamma_\nu}{s^2}\frac{s\beta_\nu^2}{\sqrt{s^2+\beta_\nu^2}}\cos(\beta_\nu h)[(1+\cos^2(\beta_\nu h))\cos(\beta_\nu z) - \beta_\nu z\sin(\beta_\nu z)]$$
$$+\frac{1}{m-2}\sum_{\nu=1}^{\infty}\frac{\Gamma_\nu}{s^2}\left[\frac{s^3}{\sqrt{s^2+\beta_\nu^2}} - s^2\right]\cos(\beta_\nu h)$$
$$\times[(m-2+m\cos^2(\beta_\nu h))\cos(\beta_\nu z) - m\beta_\nu z\sin(\beta_\nu z)]$$
$$-\sum_{n=1}^{\infty}\frac{S_n}{\sqrt{s^2+\alpha_n^2}}\left[\frac{s(s^2+\alpha_n^2)}{\sqrt{s^2+\alpha_n^2}} - \left(s^2+\frac{\alpha_n^2}{2}\right)\right]\cos(\alpha_n z)\Big\}\cos(xs)\,\mathrm{d}s$$
$$= -\frac{\sigma_0}{2G};\quad |z|<h,\quad |x|<c \tag{48}$$

which, upon using the relations (46)–(47) and interchanging the order of integration, can also be written in the form of a double integral equation i.e.,

$$\frac{1}{\pi h}\sum_{\nu=1}^{\infty}\int_{\substack{\text{crack}\\\text{faces}}}\int\{\pm v(\xi,0,\eta\}\frac{\partial^2}{\partial x\partial z}H_1[|x-\xi|;\eta,z]\,\mathrm{d}\eta\,\mathrm{d}\xi$$
$$+\frac{1}{\pi h}\sum_{n=1}^{\infty}\int_{\substack{\text{crack}\\\text{faces}}}\int\{\pm v(\xi,0,\eta)\}\frac{\partial^2}{\partial x\partial z}H_2[|x-\xi|;\eta,z]\,\mathrm{d}\eta\,\mathrm{d}\xi$$
$$-\frac{1}{\pi h}\int_{\substack{\text{crack}\\\text{faces}}}\int\{\pm v(\xi,0,\eta)\}\left(\frac{m+1}{4m}\right)\frac{\partial^2}{\partial x\partial z}\left[\frac{z}{x-\xi}\right]\mathrm{d}\eta\,\mathrm{d}\xi \tag{49}$$
$$= -\frac{\sigma_0}{2G};\ |z|\le h,\ |x|<c.$$

where

$$H_1[|x-\xi|;\eta,z] = \frac{m}{m-1}\frac{\beta_\nu h^2}{{}_eK_\nu}[\cos(\beta_\nu h)\sin(\beta_\nu z)$$
$$+\beta_\nu h\sin(\beta_\nu h)\sin(\beta_\nu z) + \beta_\nu z\cos(\beta_\nu h)\cos(\beta_\nu z)]\,\cdot$$
$$\cdot\Big\{\beta_\nu\frac{|x-\xi|}{(x-\xi)}K_1[\beta_\nu|x-\xi|][\beta_\nu h\sin(\beta_\nu h)\cos(\beta_\nu\eta) - \beta_\nu\eta\cos(\beta_\nu h)\sin(\beta_\nu\eta)$$
$$+2\left(\frac{m-1}{m}\right)\cos(\beta_\nu h)\cos(\beta_\nu\eta)]$$
$$+\beta_\nu^2\int_0^{x-\xi}K_0[\beta_\nu|x'|]\,dx'\cdot[\beta_\nu h\sin(\beta_\nu h)\cos(\beta_\nu\eta) - \beta_\nu\eta\cos(\beta_\nu h)\sin(\beta_\nu\eta)$$
$$+2\cos(\beta_\nu h)\cos(\beta_\nu\eta)]\Big\} \tag{50}$$
$$+\frac{m}{m-1}\frac{1}{\beta_{\nu e}K_\nu}\left[\frac{m-2}{m}\cos(\beta_\nu h)\sin(\beta_\nu z) + \beta_\nu h\sin(\beta_\nu h)\right.$$
$$\left.\times\sin(\beta_\nu z) + \beta_\nu z\cos(\beta_\nu h)\cos(\beta_\nu z)\right]\cdot\Big\{\Big[-\frac{\beta_\nu^2 h^2}{x-\xi}K_0[\beta_\nu|x-\xi|]$$
$$-\beta_\nu^3 h^2\frac{|x-\xi|}{(x-\xi)}K_1[\beta_\nu|x-\xi|] - \frac{2\beta_\nu h^2}{(x-\xi)|x-\xi|}K_1(\beta_\nu|x-\xi|)$$
$$+\frac{2h^2}{(x-\xi)^3}\Big]\cdot[\beta_\nu h\sin(\beta_\nu h)\cos(\beta_\nu\eta) - \beta_\nu\eta\cos(\beta_\nu h)\sin(\beta_\nu\eta)$$
$$+2\left(\frac{m-1}{m}\right)\cos(\beta_\nu h)\cos(\beta_\nu\eta)] + \Big[\beta_\nu^3 h^2\frac{|x-\xi|}{(x-\xi)}K_1[\beta_\nu|x-\xi|]$$
$$-\frac{\beta_\nu^2 h^2}{x-\xi}\Big][\beta_\nu h\sin(\beta_\nu h)\cos(\beta_\nu\eta) - \beta_\nu\eta\cos(\beta_\nu h)\sin(\beta_\nu\eta) + 2\cos(\beta_\nu h)\cos(\beta_\nu\eta)]\Big\}$$

and

$$H_2[|x-\xi|;\eta,z] = 2\frac{1}{\alpha_n^3}\left\{-\frac{\alpha_n^2}{x-\xi}K_0[\alpha_n|x-\xi|] - \frac{2\alpha_n}{|x-\xi|(x-\xi)}K_1[\alpha_n|x-\xi|] + \frac{2}{(x-\xi)^3} - \frac{\alpha_n^2}{2(x-\xi)}\right\}\cos(\alpha_n\eta)\sin(\alpha_n z). \tag{51}$$

Finally, integrating once with respect to x and z one finds*

$$\begin{aligned} &\frac{1}{\pi h}\sum_{\nu=1}^{\infty}\int_{\substack{\text{crack}\\\text{faces}}}\!\!\int \{\pm v(\xi,0,\eta)\}H_1[|x-\xi|;\eta,z]\,d\eta\,d\xi \\ &\quad+\frac{1}{\pi h}\sum_{n=1}^{\infty}\int_{\substack{\text{crack}\\\text{faces}}}\!\!\int \{\pm v(\xi,0,\eta)\}H_2[|x-\xi|;\eta,z]\,d\eta\,d\xi \\ &\quad-\frac{1}{\pi h}\int_{\substack{\text{crack}\\\text{faces}}}\!\!\int \{\pm v(\xi,0,\eta)\}\left(\frac{m+1}{4m}\right)\left[\frac{z}{x-\xi}\right]d\eta\,d\xi \\ &= -\left(\frac{\sigma_0}{2G}\right)xz;\ |x|<c,\ |z|<h. \end{aligned} \tag{52}$$

We have reduced, therefore, the problem to that of the solution of a two-dimensional singular integral equation for the unknown function $v(\xi, 0, \eta)$. This solution will be discussed in a subsequent paper.

It is interesting to note that (49) is also applicable to planar cracks of arbitrary shape that lie on the x-z-plane and are symmetric with respect to both x and z -axes**.

Perhaps it is instructive to point out some of the advantages of the present formulation over that of [4]. These are:

(i) we are seeking the solution of one integral equation
(ii) the unknown function is real and has physical meaning
(iii) the unknown function can be related directly to experimental observations
(iv) the formulation applies to a large class of planar crack problems

6. Solution to Navier's equations

Without going into the mathematical details, we may now write the displacement functions $u^{(c)}$, $v^{(c)}$ and $w^{(c)}$ in terms of the unknown function $v\ (\xi, 0, \eta)$, for all x, y and z outside the crack

$$\begin{aligned} u^{(c)} &= \pm\frac{1}{\pi h}\int_{-c}^{c}\int_{-h}^{h} v(\xi,0,\eta)\left\{\frac{1-m}{4m}\frac{x-\xi}{(x-\xi)^2+|y|^2} - \frac{m+1}{4m}\frac{\partial}{\partial x}\left[\frac{|y|^2}{(x-\xi)^2+|y|^2}\right]\right. \\ &\quad\left.+\frac{1}{4m}\left(\frac{2h^2}{3}-z^2-\eta^2\right)\frac{\partial^2}{\partial x^2}\left[\frac{x-\xi}{(x-\xi)^2+|y|^2}\right]\right\}d\eta\,d\xi \\ &\quad\pm\frac{1}{\pi h}\int_{-c}^{c}\int_{-h}^{h} v(\xi,0,\eta)\cdot\frac{\partial}{\partial y}\left(\frac{\partial^2 N}{\partial x\,\partial y}\right)d\xi\,d\eta \\ &\quad\pm\frac{1}{\pi h}\int_{-c}^{c}\int_{-h}^{h} v(\xi,0,\eta)\cdot\frac{\partial M}{\partial x}\,d\eta\,d\xi \end{aligned} \tag{53}$$

$$v^{(c)} = \frac{1}{\pi h}\int_{-c}^{c}\int_{-h}^{h} v(\xi,0,\eta)\left\{\frac{1}{2}\frac{|y|}{(x-\xi)^2+|y|^2} + \frac{m+1}{4m}\frac{\partial}{\partial x}\left[\frac{(x-\xi)|y|}{(x-\xi)^2+|y|^2}\right] + \right.$$

* The reader should notice that the function $v(\xi, 0, \eta)$ has a $\mp$ sign also.

** The same method of solution may also be used in order to derive a much more general integral equation which applies to any arbitrary crack shape or void. This matter is currently under investigation and the results will be reported in another paper.

$$+\frac{1}{4m}\left[\frac{2h^2}{3}-z^2-\eta^2\right]\frac{\partial^2}{\partial x^2}\left[\frac{|y|}{(x-\xi)^2+|y|^2}\right]\Bigg\}\,d\eta\,d\xi$$
$$-\frac{1}{\pi h}\int_{-c}^{c}\int_{-h}^{h} v(\xi,0,\eta)\cdot\frac{\partial}{\partial x}\left(\frac{\partial^2 N}{\partial x\partial y}\right)d\eta\,d\xi$$
$$+\frac{1}{\pi h}\int_{-c}^{c}\int_{-h}^{h} v(\xi,0,\eta)\frac{\partial M}{\partial y}\,d\eta\,d\xi \tag{54}$$
$$w^{(c)}=\pm\frac{1}{\pi h}\int_{-h}^{h}\int_{-c}^{c} v(\xi,0,\eta)\cdot\frac{1}{2m}\frac{\partial}{\partial x}\left[\frac{(x-\xi)z}{(x-\xi)^2+|y|^2}\right]d\xi\,d\eta$$
$$\pm\frac{1}{\pi h}\int_{-h}^{h}\int_{-c}^{c} v(\xi,0,\eta)\frac{\partial}{\partial z}(M-\tilde{M})\,d\xi\,d\eta, \tag{55}$$

where for simplicity we have adopted the following definitions:

$$N\equiv 2\sum_{n=1}^{\infty}\frac{K_o[\alpha_n\sqrt{(x-\xi)^2+|y|^2}]}{\alpha_n^2}\cos\alpha_n z\cos\alpha_n\eta \tag{56}$$

$$M\equiv\frac{m}{m-1}\sum_{\nu=1}^{\infty}\frac{1}{{}_ek_\nu}\left\{\left(\frac{m-2}{m}+\cos^2\beta_\nu h\right)\cos\beta_\nu h\cos\beta_\nu z\right.$$
$$\left.-\beta_\nu z\cos\beta_\nu h\sin\beta_\nu z\right\}\left\{-\frac{\partial^2}{\partial x^2}K_o[\beta_\nu\sqrt{(x-\xi)^2+|y|^2}]\cdot\right.$$
$$\cdot[\beta_\nu h\sin\beta_\nu h\cos\beta_\nu\eta-\beta_\nu\eta\cos\beta_\nu h\sin\beta_\nu\eta+2\left(\frac{m-1}{m}\right)\cos\beta_\nu h\cos\beta_\nu\eta]$$
$$+\beta_\nu^2K_o[\beta_\nu\sqrt{(x-\xi)^2+|y|^2}]\,[\beta_\nu h\sin\beta_\nu h\cos\beta_\nu\eta-\beta_\nu\eta\cos\beta_\nu h\sin\beta_\nu\eta$$
$$\left.+2\cos\beta_\nu h\cos\beta_\nu\eta]\right\} \tag{57}$$

and

$$\tilde{M}\equiv 4\sum_{\nu=1}^{\infty}\frac{1}{{}_ek_\nu}\cos\beta_\nu h\cos\beta_\nu z\left\{-\frac{\partial^2}{\partial x^2}K_o[\beta_\nu\sqrt{(x-\xi)^2+|y|^2}]\right.$$
$$\cdot[\beta_\nu h\sin\beta_\nu h\cos\beta_\nu\eta-\beta_\nu\eta\cos\beta_\nu h\sin\beta_\nu\eta+2\left(\frac{m-1}{m}\right)\cos\beta_\nu h\cos\beta_\nu\eta]$$
$$+\beta_\nu^2K_o[\beta_\nu\sqrt{(x-\xi)^2+|y|^2}]\cdot[\beta_\nu h\sin\beta_\nu h\cos\beta_\nu\eta-\beta_\nu\eta\cos\beta_\nu h\sin\beta_\nu\eta$$
$$\left.+2\cos\beta_\nu h\cos\beta_\nu\eta]\right\}. \tag{58}$$

In view of the above, it appears that the solution may not be separable either in cylindrical or spherical coordinates.

Finally, one may express the total strain energy stored in the system to be:

$$W=-\frac{1}{2}\int_{-h}^{h}\int_{-c}^{c}\{(v^+-v^-)\sigma_y\}_{y=0}\,dx\,dz. \tag{59}$$

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Appendix I

To find the complete homogeneous solution of Eqn. (82) of Ref. [4], we proceed as follows.

Assume first a solution of the form

$$f^{(h)}(1+\zeta) = (1-\zeta)^{2-2/m}G(\zeta),$$

where $G(\zeta)$ is an arbitrary function of ζ. Next, substitute into homogeneous difference-differential equation to find

$$(1-\zeta)^{3-2/m}G'(\zeta) + (1+\zeta)^{3-2/m}G'(-\zeta) = 0,$$

from which one may now deduce that:

$$G(\zeta) = \sum_{n=0}^{\infty} a_{2n+1}\int_0^{\zeta}(1+\zeta)^{3-2/m}(\zeta)^{2n+1}\,\mathrm{d}\zeta + c_0$$

or

$$G(\zeta) = \sum_{n=0}^{\infty} a_{2n+1}\frac{\zeta^{2n+2}}{(2n+2)}\,{}_2F_1\Big(-3+\frac{2}{m}, 2n+2;\ 2n+3;\ -\zeta\Big)$$

Appendix II

The roots of the equation $\sin(2\beta_\nu h) = -(2\beta_\nu h)$.

The equation has an infinite number of complex roots which appear in groups of four. However, as it was pointed out in the text, for this analysis only the roots with positive real parts are pertinent and furthermore, the only real root $\beta_\nu = 0$ must be discarded. Thus, if we define the roots $\beta_2, \beta_4, \beta_6, \ldots$ to be the complex conjugates of the roots $\beta_1, \beta_3, \beta_5, \ldots$, then by setting

$$2\beta_\nu h = x_\nu + iy_\nu \quad \nu = 1,3,5\ldots$$

and using a Newton-Rampson numerical method one finds

ν	x_ν	y_ν
1	4.21239	2.25073
3	10.71254	3.10315
5	17.07337	3.55109
7	23.39836	3.85881
	etc.	

Furthermore, the asymptotic behavior of the roots for large ν, i.e., for $\nu = 15, 17, 19, \ldots$, is given by the following simple relations

$$x_\nu \simeq \left(\nu + \frac{1}{2}\right)\pi$$

$$y_\nu \simeq \cos h^{-1}\left[\left(\nu + \frac{1}{2}\right)\pi\right].$$

RÉSUMÉ

En utilisant une transformation intégrale de Fourier, on réduit le problème d'une plaque fissurée d'une épaisseur arbitraire $2h$ et soumise à une contrainte extérieure uniforme de mode I, à celui de résoudre une équation intégrale singulière à deux dimensions.