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A NOTE ON THE EFFECT OF THICKNESS IN FRACTURE

by

E.S. Folias

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1. INTRODUCTION

In the field of fracture mechanics not much theoretical work has been done in order to assess analytically what effect, if any, does the specimen thickness have on the mechanism of failure. The reason for this neglect is that three-dimensional fracture mechanics problems present formidable mathematical complexities. As a result, most of our present day fracture mechanics concepts are based on already existing two-dimensional analytical solutions. However, we all recognize the fact that 3-D analytical solutions are essential for a better understanding of this complex phenomenon.

With this in mind, the author decided to investigate this subject further, at least within the framework of linear elasticity. While he recognizes the fact that linear elasticity cannot include the non-elastic behavior of the material at the crack tip per se, nevertheless it can evince many characteristics which can help us understand better the process of fracture and thus enable us to derive more accurate design criteria.

As a practical matter, the common experimental observation of a change from ductile failure at the edge to brittle fracture at the center of a broken sheet material has so far defied analysis. Yet knowledge of this could be invaluable for a complete understanding of the interaction which exists between mixed loading modes.

In one of his recent papers [1], the author discusses the three-dimensional character of the stress distribution in a thick plate containing a through-the-thickness plane crack and under the action of a stretching load. Although the above work represents accumulative part time effort on the author's part of approximately eight years, the work is still not quite complete. The reason for this is two-fold: First the problem is enormously difficult and second insufficient research funding. Be that as it may, the author in this paper would like to put forth in writing some further information that he has acquired since the publication of reference [1], but most of all to elaborate on the physical significance of some of the results.

2. PRESENT STATUS

Reference [1] discusses the three-dimensional character of the stress distribution in a plate of an arbitrary thickness 2h and containing a through-the-thickness plane crack of length 2c. At infinity, the plate is subjected to a uniform stretching load $\bar{\sigma}_o$ (see figure 1). Without going into the mathematical details, it is found that:

(i) <u>In the very inner layers of the plate, the stresses are:</u>

$$\sigma_x^{(c)} \sim \bar{\sigma}_o \, \Lambda \left\{ \frac{1}{(1-\frac{z}{h})^{2\nu}} + \frac{1}{(1+\frac{z}{h})^{2\nu}} \right\} \sqrt{\frac{c}{2\varepsilon}} \left\{ \frac{1}{2}\cos(\frac{\phi}{2}) - \frac{1}{4}\sin\phi \, \sin(\frac{3\phi}{2}) \right\} + \ldots$$

$$\sigma_y^{(c)} \sim \bar{\sigma}_o \, \Lambda \left\{ \frac{1}{(1-\frac{z}{h})^{2\nu}} + \frac{1}{(1+\frac{z}{h})^{2\nu}} \right\} \sqrt{\frac{c}{2\varepsilon}} \left\{ \frac{1}{2}\cos(\frac{\phi}{2}) + \frac{1}{4}\sin\phi \, \sin(\frac{3\phi}{2}) \right\} + \ldots$$

$$\sigma_z^{(c)} \sim \nu\bar{\sigma}_o \, \Lambda \left\{ \frac{1}{(1-\frac{z}{h})^{2\nu}} + \frac{1}{(1+\frac{z}{h})^{2\nu}} \right\} \sqrt{\frac{c}{2\varepsilon}} \cos(\frac{\phi}{2}) + \ldots$$

$$\tau_{xy}^{(c)} \sim \bar{\sigma}_o \, \Lambda \left\{ \frac{1}{(1-\frac{z}{h})^{2\nu}} + \frac{1}{(1+\frac{z}{h})^{2\nu}} \right\} \sqrt{\frac{c}{2\varepsilon}} \left\{ \frac{1}{4}\sin\phi \, \cos(\frac{3\phi}{2}) \right\} + \ldots$$

$$\tau_{yz}^{(c)} \sim -\nu\bar{\sigma}_o \, \frac{\Lambda}{h} \left\{ \frac{1}{(1-\frac{z}{h})^{2\nu+1}} - \frac{1}{(1+\frac{z}{h})^{2\nu+1}} \right\} \sqrt{\frac{c\varepsilon}{2}} \left\{ \frac{1}{2}\sin\phi \, \cos(\frac{\phi}{2}) \right\} + \ldots$$

$$\tau_{xz}^{(c)} \sim \nu\bar{\sigma}_0 \frac{\Lambda}{h} \left\{ \frac{1}{(1-\frac{z}{h})^{2\nu+1}} - \frac{1}{(1+\frac{z}{h})^{2\nu+1}} \right\} \sqrt{\frac{c\varepsilon}{2}} \left\{ (1-2\nu)\cos(\frac{\phi}{2}) + \frac{1}{2}\sin\phi\,\sin(\frac{\phi}{2}) \right\} + \ldots$$

Similarly, the corresponding displacements are:

$$u^{(c)} \sim \bar{\sigma}_0 \frac{\Lambda}{2G} \left\{ \frac{1}{(1-\frac{z}{h})^{2\nu}} + \frac{1}{(1+\frac{z}{h})^{2\nu}} \right\} \sqrt{\frac{c\varepsilon}{2}} \left\{ \frac{m-2}{m}\cos(\frac{\phi}{2}) + \frac{1}{2}\sin\phi\,\sin(\frac{\phi}{2}) \right\} + \ldots$$

$$v^{(c)} \sim \bar{\sigma}_0 \frac{\Lambda}{2G} \left\{ \frac{1}{(1-\frac{z}{h})^{2\nu}} + \frac{1}{(1+\frac{z}{h})^{2\nu}} \right\} \sqrt{\frac{c\varepsilon}{2}} \left\{ 2(\frac{m-1}{m})\sin(\frac{\phi}{2}) - \frac{1}{2}\sin\phi\,\cos(\frac{\phi}{2}) \right\} + \ldots$$

$$w^{(c)} \sim 0 + \ldots$$

where ε and ϕ are the usual cylindrical coordinates and Λ is a function of the crack to thickness and Poisson's ratios and is given by Figure 2.

In view of these results, therefore, one may make the following remarks which are applicable to the very inner* layers of the plate:

(1) The stresses possess the usual singularity.

(2) The stresses posses the usual angular distribution.

(3) The stress intensity factor K is a function of z.

(4) Exact plane strain conditions exist only on the plane $z = 0$.

(5) However, a pseudo plane strain state exists and the equation

$$\sigma_z = \nu(\sigma_x + \sigma_y)$$

is satisfied.

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*That is the immediate layers to the plane $z = 0$.

(6) As $h \to \infty$, the plane strain solution is recovered.

(7) As $\nu \to 0$, the plane stress solution is recovered.

(8) The crack opening displacement is

$$v\Big|_{y=0} \simeq \pm \left(\frac{\sigma^\circ}{2G}\right)(1-\nu)\,\Lambda \left\{ \frac{1}{(1-\frac{z}{h})^{2\nu}} + \frac{1}{(1+\frac{z}{h})^{2\nu}} \right\} \sqrt{c^2 - x^2} \;,$$

suggesting* that the crack initiates at the center ("pop-in" or "tunnel like" phenomenon). This phenomenon has also been observed experimentally [2].

(ii) <u>In the outer layers of the plate**</u>

$$\text{\underline{all} stresses:} \qquad \sigma_{ij} \sim \rho^{-\frac{1}{2}-2\nu} \frac{f_{ij}(\theta,\phi)}{\sqrt{\sin\theta}} + \ldots$$

$$\text{\underline{all} displacements:} \qquad u_i \sim \rho^{\frac{1}{2}-2\nu} \frac{g_i(\theta,\phi)}{\sqrt{\sin\theta}} + \ldots$$

where ρ, θ, ϕ stand for the usual spherical coordinates, and the functions f_{ij} and g_i are free of singularities.

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*To verify this, one must assume that the crack advances an infinitesimal distance Δc, which could be a function of x, y and z, and then resolve the 3-D problem by perturbation for the arbitrary shape Δc. A preliminary investigation showed that this is possible.

**Initially, one is tempted to write the inner expansion in terms of the spherical coordinates, for example, the stress

$$\sigma_z^{(c)} \sim \nu\bar{\sigma}_0\,\Lambda \frac{h^{2\nu}\sqrt{c}}{\sqrt{2\rho}^{\,1/2+2\nu}} \frac{1}{\cos^{2\nu}\theta\sqrt{\sin\theta}} \cos\left(\frac{\phi}{2}\right) + \ldots$$

A closer inspection, however, reveals that the above stress becomes infinite as $\theta \to \frac{\pi}{2}$. This of course contradicts one of the boundary conditions and immediately suggests that something is wrong with the solution. But this is not quite true. A careful investigation shows that other terms also contribute to the same order of singularity and consequently must be accounted for. Without going into the details (see reference 1) it can be shown that

$$\sigma_z^{(c)} = \nu\bar{\sigma}_0 \Lambda \sqrt{\frac{c}{2}} \frac{h^{2\nu}}{\rho^{1/2+2\nu}} \frac{\cos\theta \cos[(2\nu+1)\theta]}{\sqrt{\sin\theta}} \cos\left(\frac{\phi}{2}\right)$$

$$- \frac{(1-\nu)}{2(1-2\nu)} (h - z) \frac{\partial^5}{\partial z^5} \iint {}^{(0)}I_- dx dx + \ldots + R_n$$

where R_n stands for other terms of a lesser order and

$$^{(\circ)}I_\pm \equiv -\frac{C_0}{2}\sqrt{\frac{c}{2\varepsilon}}\, h^{2\nu-2} \cos\left(\frac{\phi}{2}\right) \left\{ [h \pm z - i\varepsilon]^{2-2\nu} + [h \pm z + i\varepsilon]^{2-2\nu} \right\} + R_n$$

In view of these results, one may now make the following remarks which are valid in the outer layers, i.e. the neighborhood of the point where the crack front meets the free surface of the plate:

(1) For Poisson's ratios greater than 1/4 the displacements become singular.

(2) The strength of the singularity, however, is such that the local strain energy is finite for all Poisson's ratios.*

(3) Mathematically, stress fields with such type of singularities are admissible.**

(4) Physically, linear elasticity is inadequate*** in predicting the actual behavior of the material at such corner points.

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*Consider a hemisphere with center the corner point $z = h$. The strain energy now becomes

$$W_{local} \sim \int_V \sigma_{ij}^2 \, dv \sim (1-2\nu)^2 \int_0^\rho \{\rho^{-1-4\nu} + \ldots\} \rho^2 \, d\rho = \frac{(1-2\nu)}{2} \rho^{2-4\nu} + \ldots$$

Thus $W_{local} \to 0$ as $\rho \to 0$.

**The uniqueness proof was given by Professor Calvin Wilcox, Department of Mathematics, University of Utah. For a sketch of this proof, see Appendix.

***This, however, does not imply that the three-dimensional, linear elastic results have no practical value whatsoever. The physical interpretation and use of these results is discussed later.

3. GENERAL DISCUSSION

The author in this section would like to elaborate on a few points which in the past have been misinterpreted by some researchers.

(1) The enclosed uniqueness proof (see Appendix) clearly shows that solutions which satisfy the edge condition of local finite energy are admissible. Unless one can dispute this proof, one cannot dismiss the solution on the mere supposition that the displacements must be finite.

(2) The total strain energy of the system *is finite* for all values of the Poisson's ratio. For incompressible materials, it should be noted that Navier's equations do break down*. But even for this case (assuming that the same singularity prevails) the total strain energy is finite because of the factor $(1-2\nu)$ which is present**.

(3) There are no inconsistancies with regard to the satisfaction of the boundary conditions. This matter is discussed extensively in reference [1].

(4) The analysis clearly shows that the stress intensity factor K increases as one moves from the inner layers to the outer layers of the plate. Consequently K is *not*

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*See Sokolnikoff, [3] P. 79.

**See equation (127) of reference [1]. Note that $A_{\nu}^{(k)} \sim (1-2\nu) \ldots$

a realistic parameter to use for the prediction of three-dimensional fracture. Interestingly enough, Gyekenyesi and Mendelson [4], using the numerical method of lines, were able to obtain recently the same type of behavior. Furthermore, they conclude that at the corner the stress intensity factor possesses a singularity.

(5) Alblas in his investigation [5] does not discuss the upper layers of the plate. Furthermore, the author believes that it is incorrect to compare the problem of a circular hole with that of a sharp corner-edge crack.

(6) The expression for the stresses

$$\sigma_{ij}^{(c)} \sim \rho^{-1/2-2} + \ldots$$

is valid for all $\nu \neq 1/4$. When $\nu = 1/4$, the*

$$u_i^{(c)} \sim \rho^{1/2-2(1/4)} + \ldots = \rho^0 + \ldots$$

which upon differentiation yields

$$\sigma_{ij}^{(c)} \sim 0 + \ldots$$

(7) From reference [1] and in particular equation (110) it becomes clear that the stress $\sigma_z^{(c)} \to 0$ as $\theta \to \frac{\pi}{2}$ only because of the factor cos θ. The same is also true for the other two stresses $\tau_{xz}^{(c)}$ and $\tau_{yz}^{(c)}$. On the other hand, the remaining stresses do not have this factor and therefore do not vanish as $\theta \to \frac{\pi}{2}$. Consequently, K does not have to go to zero as $z \to h$.

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*See also remark on page 16.

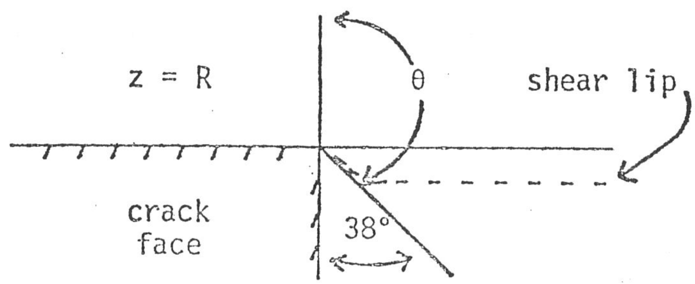
(8) The shear-lip [6] is a consequence of the fact that all stresses in the neighborhood of the corner point possess the same order of singularity.

(9) From reference [1] one may deduce that *all* stresses at the corner have at least one term of the form

$$\sigma_{ij} \sim \rho^{-1/2-2\nu} \frac{f_{ij}(\theta,\phi)}{\sqrt{\sin\theta}} + \ldots ,$$

where the functions $f_{ij}(\theta,\phi)$ are free of singularities and may be expressed in terms of definite integrals. Furthermore, one sees that the asymptotic expansion of the stresses in the inner layers of the plate represent only the first term of the series f_{ij}.

Thus, if one considers only this first term of the stresses and computes the octahedral shear stress at the corner, for $\phi = 0$ and $\nu = 1/3$, he finds that the max. value is attained at $\theta = 142°$ or at an angle of 38° from the crack front.



Experimental results obtained by Mr. F. Foreman at the Space Center NASA given values between 31° and 32°. The author believes that the reason for this fairly good agreement is due to two facts: (i) all stresses have the same approximation and (ii) the remaining terms of the series become extremely important primarily for angles $\theta = \pi/2 + \theta^*$, where $\theta^* < 20$. Of course, the latter is only a conjecture on the author's part.

(10) If now one computes after deformation the displacement function v for the very neighboring layers of the plane $z = 0$ and for $|x| < c$, he finds that

$$v^{(c)}\Big|_{y=0} \sim \pm \left(\frac{\sigma_0}{2G}\right) \Lambda\ (1-\nu) \left\{ \frac{1}{(1-\frac{z}{h})^{2\nu}} + \frac{1}{(1+\frac{z}{h})^{2\nu}} \right\} \sqrt{c^2 - x^2} + \ldots,$$

which represents a family of ellipses. Furthermore, on the plane $z = 0$, the ellipse is the sharpest and as one moves to the upper and lower neighboring layers the ellipses become progressively less sharp. This suggests, therefore, that the crack (assuming that $h > h_{crit}$ where slant fracture only prevails) initiates at the center first and assumes a bell-shaped form. This phenomenon has also been observed experimentally and is known as the "pop in" or "tunnel like" effect [2].

(11) Exact plane strain conditions exist only on the plane $z = 0$. However, all along the inner layers a "pseudo plane strain" state prevails and the equation

$$\sigma_z = \nu(\sigma_x + \sigma_y)$$

holds.

(12) The author's theoretical results are based on linear elastic theory and on a 3-D mathematical crack. In most experimental works, a crack is in general machined in the specimen and subsequently fatigued in order to come closer to the shape of a mathematical crack. But the

mere process of fatigue most likely smooths out the 90° corner and such a comparison of stresses could be meaningless. Moreover, the theoretical results show that linear elasticity is inadequate in predicting the actual behavior of the material at such corner points and that non-linear effects do take over. Therefore, the "complementary" problems corresponding to the experiments and theory are not comparable. This is because, in the former case, the principle of superposition is no longer valid.

While undoubtedly the experimental results of Villarreal, Sih and Hartranft [7] represent a substantial contribution to the field of experimental fracture mechanics, one may not use them to disprove the author's results for reasons stated above.

Be that as it may, the material, due to the presence of the high stresses in the vicinity of the corner point, yields and a plastic region (shear lip) is created. Theoretical results based on linear elastic theory can give us a good approximation of the shape and volume of the shear lip region but cannot tell us what the stress values are at any point within.

(13) Finally, a comment on Benthem's work [8]. His analysis shows that the stress intensity factor decreases as one approaches the outer layers of the plate. His results, therefore, constitute a contradiction to those of reference [1].

In general, Benthem's work is a substantial contribution to the field of fracture mechanics. His clever manipulation in constructing solutions in terms of Legendre functions is most ingenious. However, the author has some reservations concerning the validity of his results. Is the solution really separable*, particularly in θ and ϕ? Should the numerical determination of the singularity be trusted?

It becomes evident, therefore, that a third, independent, analytical solution is most desirable. Moreover, the question of completeness for both methods of solution (i.e. reference 1 and 8) is no longer an academic question but a practical necessity.

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* See remark on p. 16.

4. FUTURE RESEARCH

As it was pointed out previously, considerable more research must be carried out in order to fully understand the phenomenon of three-dimensional fracture. The following list represents only a few of the immediate goals that should be pursued for the completion of this study:

(1) An *independent* corner analysis for the verification of the corner singularity.

(2) The explicit and complete determination of the stress distribution in the vicinity of the crack throughout the thickness.

(3) The complete determination of the shape and volume of the shear lip envelope*.

(4) The exact numerical evaluation** of the stress coefficient Λ for various crack to thickness and Poisson's ratios.

(5) Reference [1] gives the total strain energy of the system as

$$W = \frac{\pi(1-\nu^2)\ \sigma_0^2\ c^2\ 2h}{E}\ F(\nu,\ c/h),$$

*A knowledge of this will determine that portion of the fractures surface where "slant fracture" prevails. Furthermore, it will give us a very good estimate of the "critical thickness," h_c, below which only shear fracture prevails.

**See Footnote 9 of Reference [1].

where F is expressed as a double series. We must seek the complete numerical evaluation of the function F.

5. A FEW FURTHER REMARKS

Reference [1] gives the stresses at every point in the plate in terms of Fourier Integral representations, which are yet to be evaluated explicity, at least within the immediate vicinity of the crack tip. Moreover, the asymptotic expansions (90) - (95), in retrospect, are valid only in the immediate layers of the middle plane. However, in order to find the complete stress distribution for all $|z| \leqslant h$, it is essential that one solves the complete system defined by the two equations (72) and 75 (a,b,c).

For example, the complete solution of the difference-differential equation (82), for all $|\zeta| \leqslant 1$. has now been found to be*

$$f(1+\zeta) + f(1-\zeta) = C_0 \left\{ (1-\zeta)^{2-2\nu} + (1+\zeta)^{2-2\nu} \right\}$$

$$+ (1-\zeta^2)^{2-2\nu} \sum_{n=0}^{\infty} \zeta^{2n+2} \int_0^1 \left[\frac{(1+\zeta\xi)^{3-2\nu}}{(1+\zeta)^{2-2\nu}} + \frac{(1-\zeta\xi)^{3-2\nu}}{(1-\zeta)^{2-2\nu}} \right]$$

$$(1-\zeta^2\xi^2)^{\gamma}\xi^{2n+1} \cdot \left[a_{2n+1} + b_{2n+1} \ln(1-\zeta^2\xi^2) \right] d\xi + \text{const.}$$

$$+ f_p(1+\zeta) + f_p(1-\zeta),$$

where f_p stands for the particular solution and where the unknown coefficients $C_0, \gamma, a_{2n+1}, b_{2n+1}$ are to be determined so that equation (72) is also satisfied for all $|z| \leqslant h$, or $|\zeta| \leqslant 1$. Notice

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*For a proof of this see Appendix II.

that the terms leading to the previously reported singularities are also present here*.

In fact, if one uses the coefficients $A_\nu^{(o)}$, which were obtained numerically from the truncated system (78) and (79), he can conpute the function

$$J = - f'' (1 + \zeta) - f'' (1 - \zeta) = \sum_{\nu=1}^{\infty} A_\nu^{(o)} (\beta_\nu h)^2 \cos(\beta_\nu h) \cos(\beta_\nu h\zeta),$$

the graph of which for $\nu = 1/3$ and $c/h = 4$ is given by fig. 4. It is interesting to note that the function J varies only slightly in the interior portion of the plate and that it undergoes rapid variations in a boundary layer adjacent to the plate-faces. This reflects, therefore, the presence of a weak singularity at the ent points $\zeta = \pm 1$.

Finally, in the neighborhood of the corner point, the author believes that the behavior of the displacements is of the form**:

$$u_i \sim \rho^{\frac{1}{2} - 2\nu} (\sin\theta)^{-\frac{1}{2}} \left\{ f_i (\theta,\phi) + \sqrt{g_i}(\theta,\phi)\ell n\rho \right\},$$

where the f_i and g_i are functions of θ and ϕ. Moreover, these functions may *not* be separable.

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*It is possible that equation (72) may also lead us to other singularities of a lesser strength.

**The reader should note that this is consistent with the author's statement in reference [1] where it was reported that the displacements are proportional to $\rho^{\frac{1}{2}-2\nu}$.

A sketch of the Uniqueness proof as given by Prof. Calvin Wilcox, Dept. of Math., University of Utah.

APPENDIX

UNIQUENESS

Boundary Value Problem. Basic unknowns are the displacements $u_i(x)$, $i = 1,2,3$ where $x \in \Omega \subset \mathbb{R}^3$; the field equations:

$$\left.\begin{aligned} e_{ij} &= \frac{1}{2}(u_{i,j} + u_{j,i}) \\ \tau_{ij} &= C_{ijk\ell}e_{k\ell} \quad ; \; (C_{ijk\ell} = C_{k\ell ij} = C_{jik\ell} = C_{ij\ell k}) \\ \tau_{ij,j} &+ F_i = 0 \end{aligned}\right\} \text{ in } \Omega .$$

The boundary conditions are:

$$\tau_{ij}\eta_j = 0 \text{ on } \partial\Omega$$

where η_j is a unit vector normal to $\partial\Omega$.

The energy density is:

$$W = \frac{1}{2} C_{ijk\ell}e_{ij}e_{k\ell} \quad \text{(positive definite)}$$

and the total energy becomes

$$\mathcal{E}(K) = \int_K W\,dx = \text{energy in a set } K.$$

Remark: $\mathcal{E}(K) < \infty \Leftrightarrow e_{ij} \in L_2(K) \Leftrightarrow \tau_{ij} \in L_2(K)$ for $\forall\, i,j$.

Edge condition:

$$u_i \in \mathcal{H}^{loc} = \left\{ v_i : \varepsilon_{ij} \in L_2^{loc}(\overline{\Omega}) \right\}$$

Generalized boundary condition: Note that if

$$u_i \in C^2(\overline{K}) \, , \; v_i \in C^1(\overline{K})$$

then

$$\int_K \tau_{ij,j}v_i dx + \int_K \tau_{ij}v_{i,j}dx = \int_{\partial K} (\tau_{ij}\eta_j)v_i ds$$

or

$$\int_K \tau_{ij,j} v_i dx + \int_K \tau_{ij} \frac{1}{2}(v_{i,j} + v_{j,i}) dx = \int_{\partial K} (\tau_{ij} n_j) v_i ds$$

Solution with Locally Finite Energy. Defined by

$$u_i \in \mathcal{H}^{loc} \qquad (\Leftrightarrow \tau_{ij} \in L_2^{loc}(\bar{\Omega}))$$

thus

$$-\int_\Omega F_i v_i dx + \int_\Omega \tau_{ij} \frac{1}{2}(v_{i,j} + v_{j,i}) dx = 0$$

for

$$v_i \in L_2^{vox}(\bar{\Omega}) \cap \mathcal{H}^{vox}$$

where

$$\mathcal{H}^{vox} = \left\{ v_i : (v_{i,j} + v_{j,i}) \in L_2^{vox}(\bar{\Omega}) \right\}$$

(note: vox means vanishes outside a compact set).

Terminology: u_i is a solution w L.F.E. (with locally finite energy).

Remark: If $u_i \in C^2(\bar{\Omega})$ is a solution w L.F.E. and $\partial\Omega$ is smooth then u_i is a classical solution of the field equations and boundary conditions.

Regularity. $F_i \in C^1(\bar{\Omega}) \Rightarrow u_i \in H^3(\bar{\Omega} \cap K)$ where K is any compact set such that $\partial\Omega \cap K$ is smooth. Hence (Sobolev inbedding theorem) $u_i \in C^1(\bar{\Omega} \cap K)$, u_i satisfies the field equations in Ω and satisfies the boundary condition on the smooth portions of $\partial\Omega$.

We now define a function $\phi(x)$ such that (see figure 3)

$$\phi(x) = \phi(r) = \begin{cases} 1 & \text{for} \quad 0 \le r \le R - \delta \\ 0 & \text{for} \quad r \ge R \\ \text{with a smooth transition across } \delta \end{cases}$$

and

$$\phi(r) \in C^\infty, \; 0 \le \phi(r) \le 1 .$$

Consider now $F_i = 0$.

We take first

$$v_i = \phi u_i \in L_2^{\text{vox}} \cap \mathcal{H}^{\text{vox}}$$

then

$$\int_{\Omega_R} \tau_{ij} \frac{1}{2} \left\{ (\phi u_i)_{,j} + (\phi u_j)_{,i} \right\} dx = 0 .$$

We take second

$$v_i = (1 - \phi) u_i$$

and integrate over Ω_R. On $\Omega_R \cap \text{supp}\,(1 - \phi)$, u_i is smooth so

$$\int_{\Omega_R} \tau_{ij,j} (1-\phi) u_i \, dx + \int_{\Omega_R} \tau_{ij} \frac{1}{2} \left\{ ((1-\phi)u_i)_{,j} + ((1-\phi)u_j)_{,i} \right\} dx$$

$$= \int_{r=R} (\tau_{ij} \hat{r}_j) u_i \, ds .$$

Adding the first and the second equation one finds

$$\int_{\Omega_R} \tau_{ij} e_{ij} dx = \int_{r=R} (\tau_{ij} \hat{r}_j) u_i ds$$

or

$$2 \int_{\Omega_R} W \, dx = \int_{r=R} (\tau_{ij} \hat{r}_j) u_i ds$$

if now we choose $u_i = O(1)$ and $\tau_{ij} \hat{r}_j = o(R)$ as $R \to \infty$ then the second integral vanishes and

$$\lim_{R \to \infty} \int_{\Omega_R} W \, dx = 0.$$

Hence the solution is unique.

Theorem Given a plate of an arbitrary thickness 2h and containing a finite crack (see fig. 1). If a solution to the 3-D Naviers equations and the corresponding boundary conditions satisfies the edge condition of local finite energy, then the solution is unique.

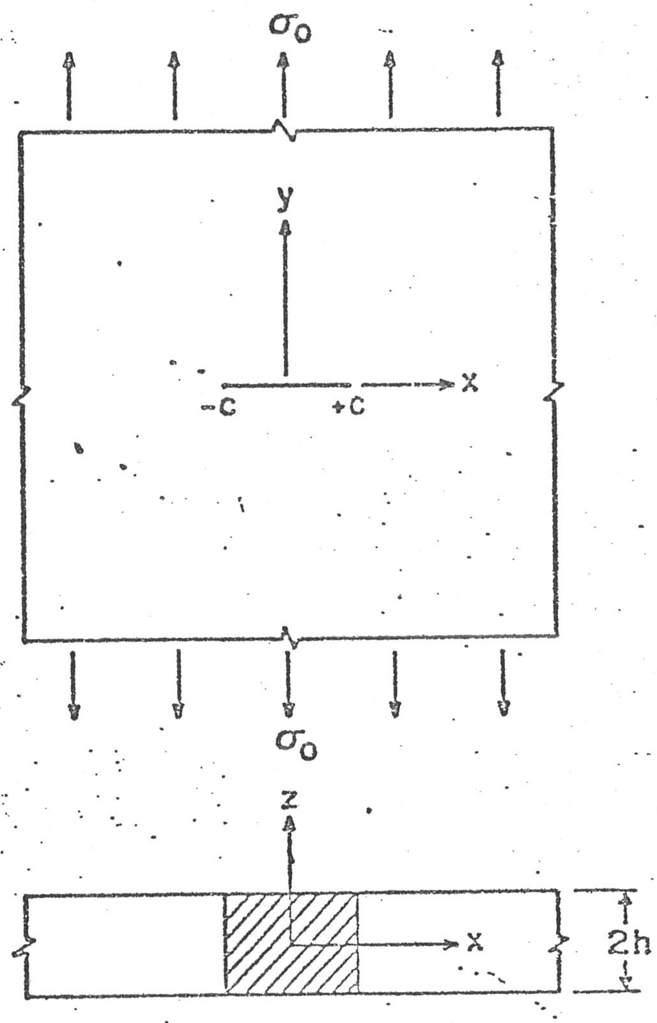

fig. 1.