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On the fracture of highway pavements

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ABSTRACT

Using a model of a single layered foundation to describe a highway or an airfield pavement and an integral formulation, the problem of a pavement containing on-the-top layer a crack of finite length 2c is solved for the stress distribution around the crack tip. The analysis shows that the stresses possess the usual $1/\varepsilon^{\frac{1}{2}}$ singular behavior which is characteristic of crack problems. Furthermore, it is found that the stress intensity factor decreases as the magnitudes of the two foundation parameters increase. Finally, as the foundation presents more resistance to shear deformation, the critical load for crack initiation increases.

Nomenclature

$2c$	= crack length
D	$= Eh^3/[12(1-\nu^2)]$ = flexural rigidity of a plate
E, E_s	= Young's modulus of plate and elastic foundation, respectively
G	= shear modulus of plate
ν, ν_s	= Poisson's ratio of plate and elastic foundation, respectively
γ_p, γ_s	= specific weights of plate and elastic foundation, respectively
h	= thickness of the plate
H	= thickness of the foundation
$\psi(z)$	= the vertical displacement function of the elastic foundation and $\psi(0)=1$
g	= gravitational acceleration
$q(x, y)$	= lateral load
p_0	= load intensity of the concentrated loads
$V_y^{(c)}, V_y^{(p)}$	= shear forces as defined in text
$M_y^{(c)}, M_y^{(p)}$	= bending moments as defined in text
$W(x, y)$	= transverse deflection of a plate in bending
$W^{(p)}(x, y)$	= transverse deflection of a continuous plate which has not been weakened by the crack
$W^{(c)}(x, y)$	= transverse deflection of a plate involving perturbations induced by the presence of the crack
$W_+^{(c)}$	$= \lim_{y\to 0^+} W^{(c)}(x, y)$, $W_-^{(c)} = \lim_{y\to 0^-} W^{(c)}(x, y)$
$r^*, \delta^*, m^*, r, \delta$	= generalized elastic characteristic constants as defined in text
X, Y, Z	= rectangular coordinates in middle plane of a plate
$x \equiv X/c,\ y \equiv Y/c,\ z \equiv Z/c$	
γ	= 0.5772 Euler's constant
$\varepsilon, \theta;\ \varepsilon e^{i\theta}$	$= x+1+iy$
ζ	$\equiv x-\xi$
ν_0	$\equiv 1-\nu$
$\sigma_x, \sigma_y, \tau_{xy}$	= stress components
$\sigma_x^{(c)}, \sigma_y^{(c)}, \tau_{xy}^{(c)}$	= perturbation stress components due to the presence of the crack
$\bar{\sigma}_b$	$\equiv 6Dm_0/h^2c^2$
P_b	= stress coefficient

λ_+^2	$= r^2+(r^4-\delta^4)^{\frac{1}{2}}$
λ_-^2	$\equiv r^2-(r^4-\delta^4)^{\frac{1}{2}}$
k	$\equiv r/\delta$

1. Introduction

It is well known that cracking of asphalt pavements is due primarily to temperature variations and to the constant application of heavy repetitive loads. Initially the effects of the cracks and holes on the riding quality are minor; however, intrusion of water very quickly causes appreciable swelling of certain subgrades with resulting bumps and a very rough riding pavement. On sandy subgrades, for example, due to material falling into the crack, severe dips can occur. Furthermore, the high stress concentrations at the tip of the crack coupled with the frost heave can result in additional cracking and spalling, with pot holes as the final result.

Consequently, considerable maintenance is usually required for crack filling and repairs. However, crack filling often has to be repeated annually, for the sealing materials which are used cannot adequately accommodate seasonal and long term contraction. Cracking, therefore, is considered to be economically very serious and it represents a major problem that pavement designers must face.

Up to now, most of the highway pavement design methods are based largely on experience expressed in the form of correlation between soil type, material properties, temperature, traffic volume and thickness. Although these methods have in the past met with reasonable success, rapid increases in the number of heavy axle loads and variety of subgrades that must support them have outrun past experience to a great extent. It is evident, therefore, that an adequate and reliable theory becomes essential.

While it is recognized that the problem is extremely difficult due to its many parameters which are involved, nevertheless the application of the principles of modern fracture mechanics will lead to a better understanding and finally, to a practical solution of this complex phenomenon.

In this paper, the authors attempt to study only one aspect of this phenomenon, namely, the effect which the weight of a vehicle has on the propagation of an already existing, on the highway pavement, crack.

2. Generalities

Analyses of plates resting on foundations usually fall into two groups. The first group follows the well known theory of Winkler and Zimmermann [1] in which the elastic foundation is considered as a system of separate unconnected springs. Such a hypothesis simplifies considerably the analysis of structures on elastic foundations and leads frequently to incorrect results. The second group follows the theory in which one describes the physical properties of the natural foundation more accurately by the hypothesis that the foundation is an elastic isotropic semi-infinite space [2]. Here again, such a hypothesis leads to cumbersome calculations and therefore the method becomes impractical.

Recently a new theory based on Vlasov's general variational method [3] has been proposed [4]. This theory considers the elastic foundation as a single or double layer model whose properties are described by two or more generalized elastic characteristics. The advantage of this theory is that it is more accurate than the theory of Winkler and Zimmermann and simpler than the theory of the elastic semi-infinite space.

The characteristics of the fracture of pavements on a Winkler-Zimmermann foundation have been investigated and the results are reported in [5]. In this paper, the authors consider the analogous problem with a single layer foundation.

Following [4], the differential equation governing the displacement function $W(X, Y)$ of a plate resting on single layered elastic foundation is

$$\nabla^4 W - 2r^{*2}\nabla^2 W + \delta^{*4} W + m^* \frac{\partial^2 W}{\partial t^2} = \frac{q}{D} \tag{2.1}$$

with the quantities r^*, δ^* and m* as constants defined by ⋆

$$r^{*2} = \frac{E_s^*}{4(1+\nu_s^*)D}\int_0^H \psi^2(z)\mathrm{d}z \tag{2.2}$$

$$\delta^{*4} = \frac{E_s^*}{(1-\nu_s^{*2})D}\int_0^H \{\psi^1(z)\}^2\mathrm{d}z \tag{2.3}$$

$$m^* = \left(\frac{\gamma_p h}{g} + \frac{\gamma_s}{g}\int_0^H \psi^2(z)\mathrm{d}z\right)\frac{1}{D} \tag{2.4}$$

and where we have introduced the following definitions

$$E_s^* = \frac{E_s}{1-\nu_s^2} \tag{2.5}$$

$$\nu_s^* = \frac{\nu_s}{1-\nu_s} \tag{2.6}$$

The usual moment components M_X, M_Y, and M_{XY} are defined in terms of the displacement function W as:

$$M_X = -D\left[\frac{\partial^2 W}{\partial X^2} + \nu\frac{\partial^2 W}{\partial Y^2}\right] \tag{2.7}$$

$$M_Y = -D\left[\frac{\partial^2 W}{\partial Y^2} + \nu\frac{\partial^2 W}{\partial X^2}\right] \tag{2.8}$$

$$M_{XY} = -D(1-\nu)\frac{\partial^2 W}{\partial X\,\partial Y} \tag{2.9}$$

and their corresponding stress components as:

$$\sigma_{X_b} = -\frac{EZ}{(1-\nu^2)}\left[\frac{\partial^2 W}{\partial X^2} + \nu\frac{\partial^2 W}{\partial Y^2}\right] \tag{2.10}$$

$$\sigma_{Y_b} = -\frac{EZ}{(1-\nu^2)}\left[\frac{\partial^2 W}{\partial Y^2} + \nu\frac{\partial^2 W}{\partial X^2}\right] \tag{2.11}$$

$$\tau_{XY_b} = -2GZ\frac{\partial^2 W}{\partial X\,\partial Y}. \tag{2.12}$$

3. A cracked plate on a single layered foundation

Formulation of the problem

Consider an infinite elastic plate which rests on a single-layered elastic foundation and contains a finite, through the thickness, crack of length 2c. The plate is subjected to two equal concentrated lateral static loads of intensity P_0 with corresponding points of application $(0, L, -h)$ and $(0, -L, -h)$ (see Fig. 1). Furthermore, in order to simplify our mathematical complexities, we assume that $L \gg c$.

⋆ It is assumed that no horizontal displacements occur in the elastic foundation and that the vertical displacement is given by a single function $\psi(z)$. From Ref. [4], a typical function is

$$\psi(z) = \frac{\sinh[r_*(H-z)]}{\sinh[r_* H]}$$

where r_* is a coefficient determining the variation with depth of the displacements.

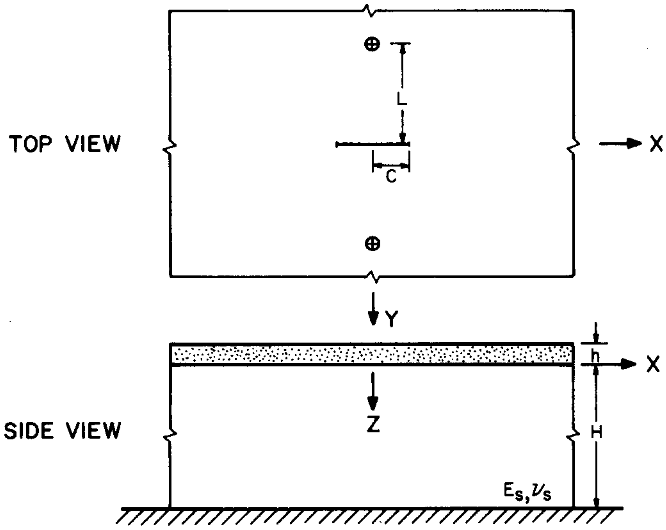

Figure 1. A cracked plate on a single layered foundation.

It is found convenient at this stage, to introduce the following dimensionless variables, i.e.,

$$x \equiv X/c, \quad y \equiv Y/c, \quad z \equiv Z/c, \quad l \equiv L/c,$$

also define

$$r = cr^*, \quad \delta = c\delta^* \tag{3.1}$$

The differential equation governing the displacement function $W(x, y)$, with x and y as dimensionless rectangular coordinates, may now be written in the form

$$\nabla^4 W - 2r^2 \nabla^2 W + \delta^4 W = c^4 \frac{P_0}{D} \delta(x)\{\delta(y-l) + \delta(y+l)\} \tag{3.2}$$

where ∇^4 and ∇^2 are respectively the biharmonic and Laplacian operators, and $\delta(\cdot)$ is the Dirac Delta function.

The boundary conditions along the crack are those of free edges. However, inasmuch as classical bending theory is used, only two boundary conditions along the crack may be satisfied. In particular, one must require that the normal moment and equivalent vertical shear vanish, i.e.,

$$\lim_{|y| \to 0} \begin{pmatrix} M_y \\ V_y \end{pmatrix} = 0 \quad \text{for} \quad -1 < x < 1 \tag{3.3}$$

In addition, it is required that the function W and all its partial derivatives up to the third order be continuous for all x and y, except for points on the segment $-1 \leqq x \leqq 1$ and $y = 0$. In order not to lose any generality, one may assume that at infinity the plate is loaded in some general manner.

Suppose now that one has already found a particular solution* satisfying (3.2) but there is a residual normal moment M_y and equivalent vertical shear V_y along the crack $|x| < 1$ of the form

$$M_y^{(p)} = -\frac{Dm_0}{c^2} \tag{3.4}$$

$$V_y^{(p)} = 0 \tag{3.5}$$

where, for simplicity, m_0 will be taken to be a constant.**

* See Section 5 for the particular solution.

** For m_0 non-constant, also see remarks in Section 5.

Mathematical statement of the problem

Assuming therefore that a particular solution has been found, we need to find a function $W^{(c)}(x,y)$ such that it satisfies the homogeneous part of the partial differential equation (3.2) and the following boundary conditions:

at $y=0$ and $|x|<1$

$$M_y^{(c)}(x,0) = -\frac{D}{c^2}\left[\frac{\partial^2 W^{(c)}}{\partial y^2} + \nu\frac{\partial^2 W^{(c)}}{\partial x^2}\right]_{|y|=0} = \frac{Dm_0}{c^2} \tag{3.6}$$

and

$$V_y^{(c)}(x,0) = -\frac{D}{c^3}\left[\frac{\partial^3 W^{(c)}}{\partial y^3} + (2-\nu)\frac{\partial^3 W^{(c)}}{\partial x^2 \partial y}\right]_{|y|=0} = 0 \tag{3.7}$$

at $y=0$ and $|x|>1$

$$\lim_{|y|\to 0}\left[\frac{\partial^n}{\partial y^n}(W_+^{(c)}) - \frac{\partial^n}{\partial y^n}(W_-^{(c)})\right] = 0 \qquad (n=0,1,2,3) \tag{3.8}$$

To complete the formulation of the problem, we require that the displacement function $W^{(c)}(x,y)$ together with its first partial derivatives be finite at infinity. These restrictions guarantee finite displacements and stresses far away from the crack.

4. Construction of the solution

Method of solution

We construct the following integral representation which has the proper behavior at infinity

$$W^{(c)}(x,y^{\pm}) = \int_0^\infty \{P_1 \exp[-(s^2+\lambda_+^2)^{\frac{1}{2}}|y|] + P_2 \exp[-(s^2+\lambda_-^2)^{\frac{1}{2}}|y|]\}\cos(xs)\mathrm{d}s \tag{4.1}$$

where P_1 and P_2 are arbitrary functions of s to be determined from the boundary conditions and the $\pm$ signs refer to $y>0$ and $y<0$, respectively.

Assuming therefore that one can differentiate under the integral sign, one finds by substituting Eqn. (4.1) into (3.7) that

$$\mp\int_0^\infty \{[\alpha^2-(2-\nu)s^2]\alpha P_1(s) + [\beta^2-(2-\nu)s^2]\beta P_2(s)\}\cos(xs)\mathrm{d}s = 0 \tag{4.2}$$

where for simplicity we let

$$\alpha \equiv (s^2+\lambda_+^2)^{\frac{1}{2}} \tag{4.3}$$

$$\beta \equiv (s^2+\lambda_-^2)^{\frac{1}{2}} \tag{4.4}$$

Equation (4.2) may be satisfied if one chooses

$$P_1(s) = [\beta^2-(2-\nu)s^2]\beta P(s) \tag{4.5}$$

$$P_2(s) = -[\alpha^2-(2-\nu)s^2]\alpha P(s) \tag{4.6}$$

where the function $P(s)$ is still largely arbitrary. Similarly, substituting (4.1) into (3.6) and utilizing Eqns. (4.5) and (4.6) one obtains

$$-\int_0^\infty \{(\alpha^2-\nu s^2)[\beta^2-(2-\nu)s^2]\beta - (\beta^2-\nu s^2)[\alpha^2-(2-\nu)s^2]\alpha\}p(s)\cos(xs)\mathrm{d}s = m_0\,, \quad \text{for } |x|<1 \tag{4.7}$$

Next, it can easily be shown that all the continuity conditions may be satisfied if one considers the following expression to vanish

$$\int_0^\infty \alpha\beta(\alpha^2-\beta^2)p(s)\cos(xs)\mathrm{d}s = 0 \quad \text{for} \quad |x|>1 \tag{4.8}$$

Reduction to single integral equation

Because we are unable to solve dual integral equations of the type discussed in the previous section, we will therefore reduce the problem to that of the solution of a singular integral equation. If one now lets

$$u(x) = \int_0^\infty \alpha\beta(\alpha^2-\beta^2)p(s)\cos(xs)\mathrm{d}s \quad \text{for} \quad |x|<1 \tag{4.9}$$

Then by Fourier inversion

$$P(s) = \frac{2}{\pi\alpha\beta(\alpha^2-\beta^2)}\int_0^1 u(\xi)\cos(s\xi)\mathrm{d}\xi \tag{4.10}$$

where the function $u(\xi)$, due to the symmetry of the problem is an even function. Thus, formally, substituting (4.10) into (4.7) one, after changing the order of integration and rearranging, has

$$\int_{-1}^1 L(\lambda_+,\lambda_-,|x-\xi|)u(\xi)\mathrm{d}\xi = -m_0\pi x \quad \text{for} \quad |x|<1 \tag{4.11}$$

where the kernel $L(\lambda_+,\lambda_-,|x-\xi|)$ is given by the expression

$$L(\lambda_+,\lambda_-,|x-\xi|) \equiv \int_0^\infty \left\{\frac{(\alpha^2-\nu s^2)[\beta^2-(2-\nu)s^2]}{s\alpha(\alpha^2-\beta^2)} - \frac{(\beta^2-\nu s^2)[\alpha^2-(2-\nu)s^2]}{s\beta(\alpha^2-\beta^2)}\right\}\sin(x-\xi)s\,\mathrm{d}s \tag{4.12}$$

whose asymptotic form for small λ_+, λ_-'s is

$$\begin{aligned} L(\lambda_+,\lambda_-,|\zeta|) = (1-\nu)(3+\nu)\frac{1}{\zeta} + \Big\{ &-\tfrac{1}{16}[\nu(2-\nu)(5-12\gamma+12\ln 2) \\ &+(3-4\gamma+4\ln 2)-8\nu(1-2\gamma+2\ln 2)](\lambda_+^2+\lambda_-^2) \\ &+\tfrac{1}{4}(3\nu+1)(1-\nu)\frac{\lambda_+^4\ln\lambda_+-\lambda_-^4\ln\lambda_-}{\lambda_+^2-\lambda_-^2} + (1-\nu)\frac{\lambda_+^2\lambda_-^2}{\lambda_+^2-\lambda_-^2}\ln\frac{\lambda_+}{\lambda_-}\Big\}\zeta \\ &+\tfrac{1}{4}(1+3\nu)(1-\nu)(\lambda_+^2+\lambda_-^2)\zeta\ln\zeta + \ldots \end{aligned} \tag{4.13}$$

We require now that the solution $u(\xi)$ be Hölder continuous for some positive Hölder index μ for all x in the closed interval $[-1, 1]$. In particular, $u(\xi)$ is to be bounded near the ends of the crack.

Solution of the integral equation for small λ_+ and λ_-

Case I. $\delta \ll r < 1$

The asymptotic form of the kernel becomes:

$$\begin{aligned} L(\lambda_+,\lambda_-,|\zeta|) = (1-\nu)(3+\nu)\frac{1}{\zeta} + \{ &-\tfrac{1}{8}[\nu(2-\nu)(5-12\gamma+12\ln 2) \\ &+(3-4\gamma+4\ln 2)-8\nu(1-2\gamma+2\ln 2)]r^2 \\ &+\tfrac{1}{4}(3\nu+1)(1-\nu)(2\ln r+\ln 2)r^2\}\zeta \\ &+\tfrac{1}{2}(1+3\nu)(1-\nu)r^2\zeta\ln\zeta + \ldots \end{aligned} \tag{4.14}$$

Thus, following the same method of solution as that described in Ref. [6], one may let

$$u(\xi) = (1-\xi^2)^{\frac{1}{2}}[A_1 + A_2 r^2(1-\xi^2) + \ldots] \; ; \qquad |\xi| < 1 \tag{4.15}$$

where the coefficients A_i's are functions of r but not of ξ.

Substituting (4.15) into (4.11) and making use of the relations given in the Appendix, find, by equating coefficients, that:

$$\begin{aligned} A_1 = m_0(12 - \tfrac{3}{2}r^2)\{\{12(1-\nu)(3+\nu) + \{-\tfrac{3}{4}[\nu(2-\nu)(5-12\gamma+12\ln 2) + \\ -2(3-4\gamma+4\ln 2) + 16\nu(1-2\gamma+2\ln 2)] + \tfrac{3}{2}(3\nu+1)(1-\nu)(1-\ln 2+2\ln r)\}r^2 + \\ + \{\tfrac{3}{64}[\nu(2-\nu)(5-12\gamma+12\ln 2) + (3-4\gamma+4\ln 2) - 8\nu(1-2\gamma+2\ln 2)] + \\ -\tfrac{3}{32}(3\nu+1)(1-\nu)(\tfrac{3}{2} - \ln 2 + 2\ln r)\}r^4\}\}^{-1} \end{aligned} \tag{4.16}$$

Case II. $r = k\delta < 1$

The asymptotic form of the kernel now becomes

$$\begin{aligned} L(\lambda_+, \lambda_-, |\zeta|) = (1-\nu)(3+\nu)\frac{1}{\zeta} + \Big\{ -\tfrac{1}{8}[\nu(2-\nu)(5-12\gamma+12\ln 2) + \\ +(3-4\gamma+4\ln 2) - 8\nu(1-2\gamma+2\ln 2)]k^2\delta^2 + \\ +\tfrac{1}{4}(3\nu+1)(1-\nu)\left[\frac{2k^4-1}{4(k^4-1)^{\frac{1}{2}}}\ln\frac{k^2+(k^4-1)^{\frac{1}{2}}}{k^2-(k^4-1)^{\frac{1}{2}}} + 2k^2\ln\delta\right]\delta^2 + \\ + \frac{(1+\nu)}{4(k^4-1)^{\frac{1}{2}}}\ln\frac{k^2+(k^4-1)^{\frac{1}{2}}}{k^2-(k^4-1)^{\frac{1}{2}}}\delta^2\Big\}\zeta + \tfrac{1}{2}(1+3\nu)(1-\nu)k^2\delta^2\cdot\zeta\ln\zeta + \ldots \end{aligned} \tag{4.17}$$

and if again we let

$$u(\xi) = (1-\xi^2)^{\frac{1}{2}}[B_1 + B_2\delta^2(1-\xi^2) + \ldots] \tag{4.18}$$

then

$$\begin{aligned} B_1 = m_0\Big\{(1-\nu)(3+\nu) + \frac{1}{8(k^4-1)^{\frac{1}{2}}}\ln\frac{k^2+(k^4-1)^{\frac{1}{2}}}{k^2-(k^4-1)^{\frac{1}{2}}}[(1+\nu)+\tfrac{1}{4}(3\nu+1)(1-\nu)(2k^4-1)]\delta^2 + \\ -\tfrac{1}{16}[\nu(2-\nu)(5-12\gamma+12\ln 2) + (3-4\gamma+4\ln 2) - 8\nu(1-2\gamma+2\ln 2) + \\ -2(1+3\nu)(1-\nu)(1-2\ln 2)]k^2\delta^2 + \tfrac{1}{4}(3\nu+1)(1-\nu)k^2(\ln\delta)\delta^2 + \\ +(1-\nu)(3+\nu)\frac{3k^2\delta^2}{4-3k^2\delta^2} \\ -\tfrac{3}{32}[\nu(2-\nu)(5-12\gamma+12\ln 2) + (3-4\gamma+4\ln 2) - 8\nu(1-2\gamma+2\ln 2) + \\ -2(1+3\nu)(1-\nu)(\tfrac{1}{2} - 2\ln 2)]\frac{k^4\delta^4}{4-3k^2\delta^2} + \\ + \frac{3}{16(k^4-1)^{\frac{1}{2}}}\ln\frac{k^2+(k^4-1)^{\frac{1}{2}}}{k^2-(k^4-1)^{\frac{1}{2}}}[(1+\nu)+\tfrac{1}{4}(3\nu+1)(1-\nu)(2k^4-1)]\frac{k^2\delta^4}{4-3k^2\delta^2} + \\ +\tfrac{3}{8}(3\nu+1)(1-\nu)\frac{k^4\delta^4}{4-3k^2\delta^2}(\ln\delta) + \ldots\Big\}^{-1} \end{aligned} \tag{4.19}$$

The displacement function W

For both of above cases, we obtain

$$P(s) = \frac{1}{(\lambda_+^2 - \lambda_-^2)(s^2+\lambda_+^2)^{\frac{1}{2}}(s^2+\lambda_-^2)^{\frac{1}{2}}}\left[D_1 \frac{J_1(s)}{s} + 3D_2 \Delta^2 \frac{J_2(s)}{s^2} + \right.$$

$$\left. + 15 D_3 \Delta^4 \frac{J_3(s)}{s^3} + \ldots \right. \tag{4.20}$$

where in Case I,

$$D_i = A_i\,, \quad i = 1, 2, 3, \ldots \quad \text{and} \quad \Delta = r \tag{4.21}$$

in Case II,

$$D_i = B_i \quad i = 1, 2, 3, \ldots \quad \text{and} \quad \Delta = \delta \tag{4.22}$$

Therefore, a substitution of (4.20), (4.5), (4.6) into (4.1) will determine the bending deflection $W^{(c)}$ as follows:

$$W^{(c)}(x, y^{\pm}) = \int_0^\infty \left\{ \left[-(1-\nu)(s^2+\lambda_+^2)^{\frac{1}{2}} + \frac{(1-\nu)\lambda_+^2 + \lambda_-^2}{(s^2+\lambda_+^2)^{\frac{1}{2}}} \right] \exp\left[-(s^2+\lambda_+^2)^{\frac{1}{2}}|y|\right] + \right.$$

$$\left. - \left[-(1-\nu)(s^2+\lambda_-^2)^{\frac{1}{2}} + \frac{(1-\nu)\lambda_-^2 + \lambda_+^2}{(s^2+\lambda_+^2)^{\frac{1}{2}}} \right] \exp\left[-(s^2+\lambda_-^2)^{\frac{1}{2}}|y|\right] \right\} \times$$

$$\times \left\{ D_1 \frac{J_1(s)}{s} + 3\, D_2 \Delta^2 \frac{J_2(s)}{s^2} + 15\, D_3 \Delta^4 \frac{J_3(s)}{3} + \ldots \right\} \frac{\cos(xs)\mathrm{d}s}{(\lambda_+^2 - \lambda_-^2)} \tag{4.23}$$

The stress field ahead of the crack tip

In view of the displacement function W, the bending stresses at the surface $z = -h/c$ may now be computed as

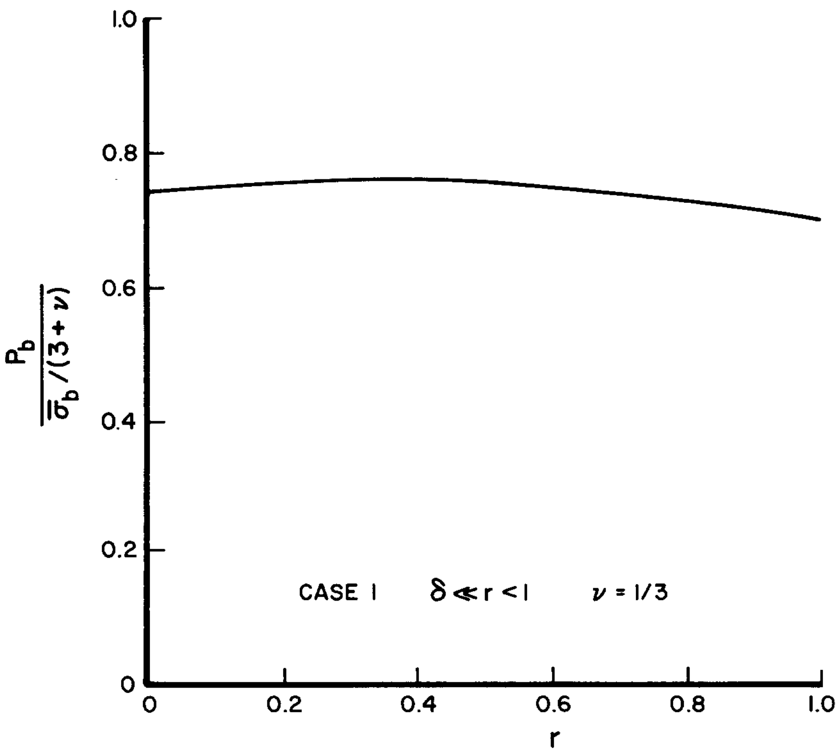

Figure 2. Stress coefficient *versus* r.

$$\sigma_x = \frac{P_b}{(2\varepsilon)^{\frac{1}{2}}}\left(-\frac{3-3\nu}{4}\cos\frac{\theta}{2}-\frac{1-\nu}{4}\cos\frac{5\theta}{2}\right)+O(\varepsilon^0) \tag{4.24}$$

$$\sigma_y = \frac{P_b}{(2\varepsilon)^{\frac{1}{2}}}\left(\frac{11+5\nu}{4}\cos\frac{\theta}{2}+\frac{1-\nu}{4}\cos\frac{5\theta}{2}\right)+O(\varepsilon^0) \tag{4.25}$$

$$\tau_{xy} = \frac{P_b}{(2\varepsilon)^{\frac{1}{2}}}\left(-\frac{7+\nu}{4}\sin\frac{\theta}{2}-\frac{1-\nu}{4}\sin\frac{5\theta}{2}\right)+O(\varepsilon^0) \tag{4.26}$$

where the stress coefficient P_b is given by
(i) for $\delta \ll r < 1$

$$\begin{aligned} P_b = {} & \frac{\bar{\sigma}_b}{(3+\nu)}(12-\tfrac{3}{2}r^2)\Big\{12+\{-\tfrac{3}{4}[\nu(2-\nu)(5-12\gamma+12\ln 2)+ \\ & -2(3-4\gamma+4\ln 2)+16\nu(1-2\gamma+2\ln 2)]+ \\ & +\tfrac{3}{2}(3\nu+1)(1-\nu)(1-\ln 2+2\ln r)\}\frac{r^2}{(1-\nu)(3+\nu)}+ \\ & +\{\tfrac{3}{64}[\nu(2-\nu)(5-12\gamma+12\ln 2)+(3-4\gamma+4\ln 2)-8\nu(1-2\gamma+2\ln 2)] \\ & -\tfrac{3}{32}(3\nu+1)(1-\nu)(\tfrac{3}{2}-\ln 2+2\ln r)\}\frac{r^4}{(1-\nu)(3+\nu)}+\ldots\Big\}^{-1} \end{aligned} \tag{4.27}$$

Notice that δ does not appear in Eqn. (4.27) for it is negligible. A plot of this is given in Fig. 2.
(ii) for $r = k\delta < 1$, where k is a real constant

$$\begin{aligned} P_b = {} & \frac{\bar{\sigma}_b}{(3+\nu)}\Big\{1+\frac{1}{8(k^4-1)^{\frac{1}{2}}}\ln\frac{k^2+(k^4-1)^{\frac{1}{2}}}{k^2-(k^4-1)^{\frac{1}{2}}}\,\frac{(1+\nu)+\frac{1}{4}(3\nu+1)(1-\nu)(2k^4-1)}{(1-\nu)(3+\nu)}\delta^2+ \\ & -\frac{1}{16(1-\nu)(3+\nu)}[\nu(2-\nu)(5-12\gamma+12\ln 2)+(3-4\gamma+4\ln 2)+ \\ & -8\nu(1-2\gamma+2\ln 2)-2(1+3\nu)(1-\nu)(1-2\ln 2)]k^2\delta^2+ \\ & +\tfrac{1}{4}k^2(\ln\delta)\delta^2+\frac{3k^2\delta^2}{4-3k^2\delta^2}-\frac{3}{32(1-\nu)(3+\nu)}[\nu(2-\nu)(5-12\gamma+12\ln 2)+ \\ & +(3-4\gamma+4\ln 2)-8\nu(1-2\gamma+2\ln 2)-2(1+3\nu)(\tfrac{1}{2}-2\ln 2)]\frac{k^4\delta^4}{4-3k^2\delta^2}+ \\ & +\frac{3}{16(k^4-1)^{\frac{1}{2}}}\ln\frac{k^2+(k^4-1)^{\frac{1}{2}}}{k^2-(k^4-1)^{\frac{1}{2}}}\,\frac{(1+\nu)+\frac{1}{4}(3\nu+1)(1-\nu)(2k^4-1)}{(1+\nu)(3+\nu)}\,\frac{k^2\delta^4}{4-3k^2\delta^2}+ \\ & +\frac{3k^4\delta^4}{32-24k^2\delta^2}\ln\delta+\ldots\Big\}^{-1} \end{aligned} \tag{4.28}$$

A plot of this is given in Fig. 3.

5. The particular solution

In general, the actual stress fields will depend upon the contributions of the particular solutions reflecting the magnitude and distribution of the applied load. On the other hand, the singular part of the solution, that is the terms producing infinite elastic stresses at the crack tip, will depend upon the local stresses existing along the locus of the crack before it is cut, which of course are precisely the stresses which must be removed by the particular solutions described above in order to obtain the stress-free edges as required physically.

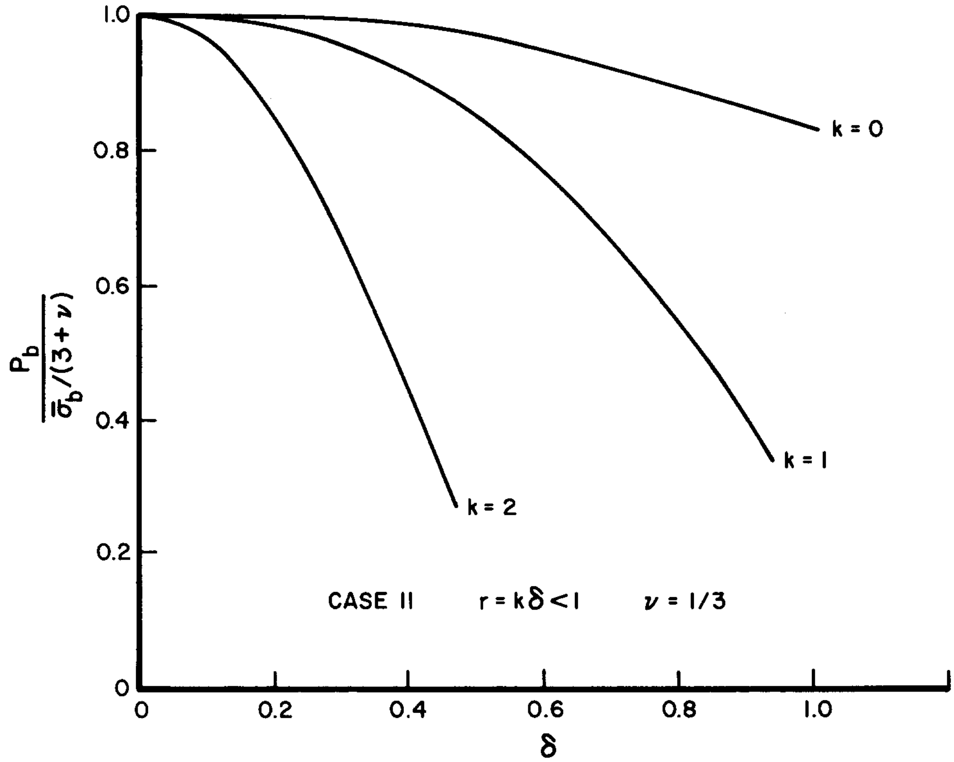

Figure 3. Stress coefficient *versus* δ.

To get the particular solution, one has to solve the following differential equation:

$$\nabla^4 W^{(p)} - 2r^2 \nabla^2 W^{(p)} + \delta^4 W^{(p)} = \frac{P_0 c^4}{D} \delta(x)\{\delta(y-l) + \delta(y+l)\} \tag{5.1}$$

with boundary conditions

$$\lim_{\substack{x\to\infty \\ y\to\infty}} M^{(p)}(x, y) = 0$$

and

$$\tag{5.2}$$

$$\lim_{\substack{x\to\infty \\ y\to\infty}} V^{(p)}(x, y) = 0 \tag{5.3}$$

Suppose one chooses the integral representation

$$W^{(p)} = \int_0^\infty R(s, y) \cos(xs) ds \tag{5.4}$$

then substitutes into (5.1), and follows the Green's function method of [12], without going into details, will find

$$W^{(p)} = -\frac{P_0 c^4}{2\pi D(\lambda_+^2 - \lambda_-^2)} \{K_0[\lambda_+(x^2 + |y-l|^2)^{\frac{1}{2}}] + K_0[\lambda_+(x^2 + |y+l|^2)^{\frac{1}{2}}] + \\ - K_0[\lambda_-(x^2 + |y-l|^2)^{\frac{1}{2}}] - K_0[\lambda_-(x^2 + |y+l|^2)^{\frac{1}{2}}]\} \tag{5.5}$$

It follows at $y=0$ and $|x| < 1$

$$M_y^{(p)}(x, 0) = -\frac{Dm_0}{c^2} \tag{5.6}$$

where

$$m_0 \equiv -\frac{P_0 c^4}{2\pi D(\lambda_+^2-\lambda_-^2)}\left\{\frac{(1-\nu)}{(x^2+l^2)^{\frac{1}{2}}}\left[1-\frac{2x^2}{x^2+l^2}\right]\{\lambda_+ K_1[\lambda_+(x^2+l^2)^{\frac{1}{2}}]+ -\lambda_- K_1[\lambda_-(x^2+l^2)^{\frac{1}{2}}]\} + \frac{l^2+\nu x^2}{x^2+l^2}\{\lambda_+^2 K_0[\lambda_+(x^2+l^2)^{\frac{1}{2}}]-\lambda_-^2 K_0[\lambda_-(x^2+l^2)^{\frac{1}{2}}]\}\right\} \tag{5.7}$$

and

$$V_y^{(p)}(x,0)=0\,. \tag{5.8}$$

Now, since we have already assumed that $1 \ll l$, it is easy to see that the above bending moment (along the crack) is approximately a constant, i.e., $-Dm_0/c^2$. Alternately, as an engineering approximation, one may think of the quantity $(-Dm_0/c^2)$ as an upper bound, or lower bound, or even a mean value of the precise bending moment along the crack in order to obtain an estimate of the stresses in the vicinity of the crack.

Furthermore, if one wants a more accurate result than that of a uniform residual moment, he must expand m_0, for $|x|<1$, in the form $\Sigma_n a_n x^{2n}$ (even power because of the symmetry of the problem), and again our previous method of solution will still be applicable. Of course, the coefficients A_n and B_n in this case may change, but the character of the solution will still remain the same.

Critical loading

Following the concepts of the Theory of Fracture Mechanics, one may make an energy balance* to derive** the following approximate fracture criterion for crack initiation.

For $r=k\delta$

$$P_{0_{\text{critical}}}=\frac{2(9-7\nu)(3+\nu)}{(33+6\nu-7\nu^2)(1+\nu)}\frac{h^2}{c^2}\sigma^*\cos^{-1}\left[\exp\left(-\frac{\pi K^2}{8\sigma^{*2}c}\right)\right]IJ \tag{5.9}$$

where K = fracture toughness (of the top layer),

$$\sigma^* \equiv \frac{\sigma_{\text{yield}}+(\sigma_{\text{yield}}+\sigma_{\text{ultimate}})/2}{2} \quad \text{(of the top layer)} \tag{5.10}$$

$$\begin{aligned} I \equiv \Bigg\{ & 1+\frac{1}{8(k^4-1)^{\frac{1}{2}}}\ln\frac{k^2+(k^4-1)^{\frac{1}{2}}}{k^2-(k^4-1)^{\frac{1}{2}}}\,\frac{(1+\nu)+\frac{1}{4}(3\nu+1)(1-\nu)(2k^4-1)}{(1-\nu)(3+\nu)}\,\delta^2+ \\ & -\frac{1}{16(1-\nu)(3+\nu)}[\nu(2-\nu)(5-12\gamma+12\ln 2)+(3-4\gamma+4\ln 2)+ \\ & -8\nu(1-2\gamma+2\ln 2)-2(1+3\nu)(1-\nu)(1-2\ln 2)]k^2\delta^2+\tfrac{1}{4}(\ln\delta)k^2\delta^2+\frac{3k^2\delta^2}{4-3k^2\delta^2} \\ & -\frac{3}{32(1-\nu)(3+\nu)}[\nu(2-\nu)(5-12\gamma+12\ln 2)+(3-4\gamma+4\ln 2) \\ & -8\nu(1-2\gamma+2\ln 2)-2(1+3\nu)(\tfrac{1}{2}-2\ln 2)]\frac{k^4\delta^4}{4-3k^2\delta^2}+ \\ & +\frac{3}{16(k^4-1)^{\frac{1}{2}}}\ln\frac{k^2+(k^4-1)^{\frac{1}{2}}}{k^2-(k^4-1)^{\frac{1}{2}}}\,\frac{(1+\nu)+\frac{1}{4}(3\nu+1)(1-\nu)(2k^4-1)}{(1+\nu)(3+\nu)}\,\frac{k^2\delta^4}{4-3k^2\delta^2} \\ & +\frac{3k^4\delta^4}{32-24k^2\delta^2}\ln\delta+\dots\Bigg\}^{-1} \end{aligned} \tag{5.11}$$

* The approach is based on a corollary of the First Law of Thermodynamics and was first applied to the phenomenon of fracture by Griffith [13].

** For more details see reference [14].

$$
\begin{aligned}
J \equiv \Bigg\{ & \frac{(1-\nu)}{2} + (1+\nu)\gamma + \frac{(1+\nu)}{4(k^4-1)^{\frac{1}{2}}}\Bigg[(k^2+(k^4-1)^{\frac{1}{2}}) \ln \frac{(k^2+(k^4-1)^{\frac{1}{2}})(\delta l)^2}{4} + \\
& -(k^2-(k^4-1)^{\frac{1}{2}}) \ln \frac{(k^2-(k^4-1)^{\frac{1}{2}})(\delta l)^2}{4} \Bigg] + \tfrac{1}{16}[4(3+\nu)\gamma - 11 - 5\nu]\, k^2 (\delta l)^2 + \\
& + \frac{(3+\nu)(\delta l)^2}{32(k^4-1)^{\frac{1}{2}}}\Bigg[(k^2+(k^4-1)^{\frac{1}{2}})^2 \ln \frac{(k^2+(k^4-1)^{\frac{1}{2}})(\delta l)^2}{4} + \\
& - (k^2-(k^4-1)^{\frac{1}{2}})^2 \ln \frac{(k^2-(k^4-1)^{\frac{1}{2}})(\delta l)^2}{4} \Bigg] + \ldots \Bigg\}^{-1}
\end{aligned}
\tag{5.12}
$$

The plots of I and J are given in Figs. 4 and 5, respectively.

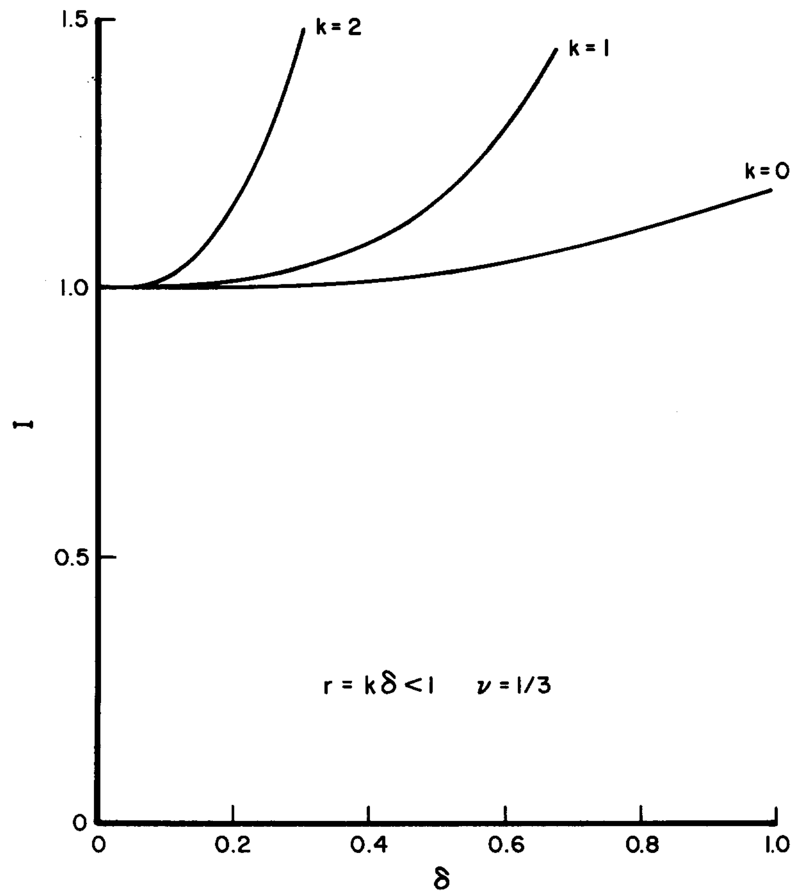

Figure 4. *I versus δ.*

6. Conclusions

The local stresses near the crack tip are found to be proportional to the usual $1/\varepsilon^{\frac{1}{2}}$ singular behavior and the usual angular distribution. Furthermore, the stress intensity factor is a function of the two characteristic elastic moduli, i.e., r and δ, and in the limit as $r \to 0$ and $\delta \to 0$ we recover the well known results of a plate without a foundation.

Typical terms of the stresses are of the form

(1) for $r = k\delta$ and δ small:

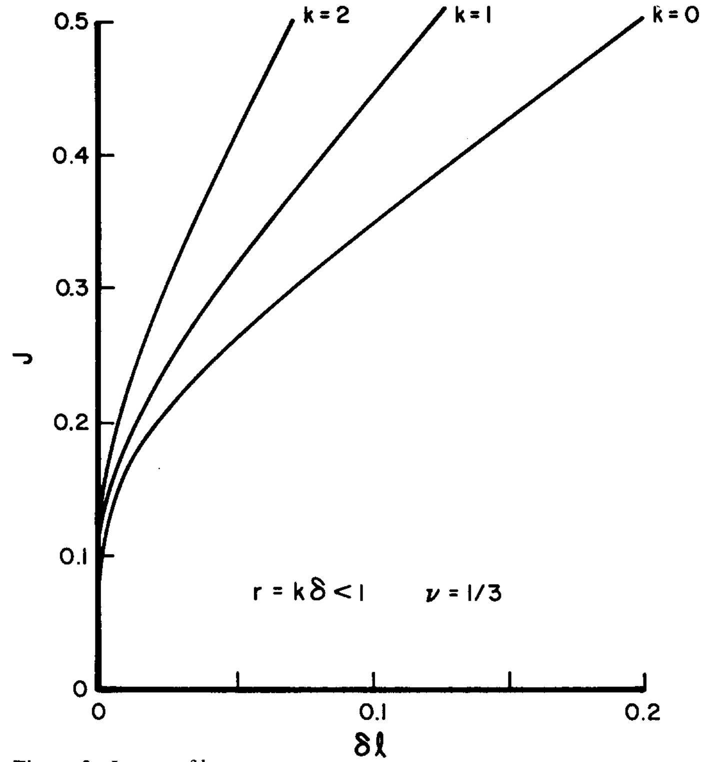

Figure 5. *J versus δl.*

$$\frac{\sigma_{\text{sup.}}}{\sigma_{\text{unsup.}}} \simeq \{1+a\delta^2+\ldots\}^{-1}$$

(ii) for $\delta \to 0$ and r small:

$$\frac{\sigma_{\text{sup.}}}{\sigma_{\text{unsup.}}} \simeq (12-\tfrac{3}{2}r^2)\{12+br^2+\ldots\}^{-1}$$

where a and b are positive constants. We conclude, therefore, that the general effect of a pavement foundation is to decrease the magnitude of the stresses in the neighborhood of the crack tip and as a result prevent further fracture. This decrease clearly depends on the values of the two parameters which characterize the foundation (see Figs. 2 and 3).

Furthermore, it is found that the critical load for crack initiation increases as the foundation presents more resistance to shear deformation.

Finally, by letting $k \to 0$, one recovers the results for a Winkler and Zimmermann foundation [5].

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RÉSUMÉ

En utilisant un modèle représentant une fondation à simple couche, pour décrire une route ou un tarmac, et une formulation par intégrales, on résoud le problème d'un revêtement comportant, dans la couche de surface, une fissure de longueur $2c$ soumise à ses extrémités à un champ de contraintes déterminé. L'analyse montre que les contraintes présentent le comportement usuel représenté par $1/\varepsilon^{\frac{1}{2}}$, qui est caractéristique des problèmes de fissurations. De plus, on trouve que le facteur d'intensité des contraintes décroît lorsque les grandeurs des deux paramètres caractérisant la fondation s'accroissent. Enfin, lorsque la fondation présente une plus grande résistance à la déformation par cisaillement, il s'ensuit un accroissement de la charge critique nécessaire à l'amorçage de la fissure.

ZUSAMMENFASSUNG

Durch Gebrauch eines Modelles von einem Fundament mit einer einzigen Schicht zur Beschreibung eines Autobahnoder Flughafenbelages, und einer Formulierung durch Integrale wird das Problem des Belages, das einen Riß von endlicher Länge $2c$ in der Oberflächenschicht enthält, in Hinsicht der Spannungsverteilung an der Rißspitze gelöst. Die Analyse ergibt daß die Spannungen das gebräuchliche $1/\varepsilon^{\frac{1}{2}}$ singuläres Benehmen enthalten was spezifisch für Rißprobleme ist. Weiterhin wurde gefunden daß die Spannungsintensitätsfaktoren abnehmen wenn die Werte der zwei Fundamentparametern zunehmen. Und schließlich da das Fundament einen größeren Widerstand gegenüber Querverformungen besitzt, nimmt die kritische Last der Rißauslösung zu.