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THE STRESSES IN A CRACKED SPHERICAL SHELL

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1. Introduction. One of the problems in fracture mechanics which apparently has not received extensive theoretical treatment is that concerning the effect of initial curvature upon the stress distribution in a thin sheet containing a crack. Considerable work has been carried out on initially flat sheets subjected to either extensional or bending stresses, and for small deformations the superposition of these separate effects [1] is permissible. On the other hand, if a thin sheet is initially curved, a bending load will generally produce both bending and extensional stresses, and similarly a stretching load will also induce both bending and extensional stresses. The subject of eventual concern therefore is that of the simultaneous stress fields produced in an initially curved sheet containing a crack.

In the following, we consider bending and stretching of thin shells of revolution, as described by traditional two-dimensional linear theory, with the additional assumption of shallowness. In speaking of the formulation of two-dimensional differential equations, we mean the transition from the exact three-dimensional elasticity problem to that of two-dimensional approximate formulation, which is appropriate in view of the "thinness" of the shell. In this paper we limit ourselves to isotropic and homogeneous shallow segments* of elastic spherical shells of constant thickness. It is furthermore assumed that the shell is subjected to small deformations and strain so that the stress-strain relations may be established through Hooke's law.

2. Formulation of the Problem. Consider a portion of a thin, shallow spherical shell of constant thickness h and subjected to an internal pressure $q(X, Y)$ (see fig. 1). The material of the shell is assumed to be homogeneous and isotropic and at the apex there exists a radial cut of length $2c$ with respect to the apex. It is convenient at this point to introduce dimensionless coordinates, namely

$$x \equiv \frac{X}{c}, \qquad y \equiv \frac{Y}{c} \tag{2.1}$$

Following Reissner [2], the coupled differential equations governing the deflection function $W(x, y)$ and the stress function $F(x, y)$, with x and y as dimensionalized rectangular coordinates of the base plane, are given by:

$$-\frac{Ehc^2}{R}\nabla^2 W + \nabla^4 F = 0, \qquad \nabla^4 W + \frac{c^2}{RD}\nabla^2 F = 0 \tag{2.2, 3}$$

As to boundary conditions, one must require that the normal moment, equivalent vertical shear, and normal and tangential in-plane stresses vanish along the crack. However, suppose that one has already found a particular solution satis-

* A segment will be called shallow if the ratio of height to base diameter is less than, say, 1/8.

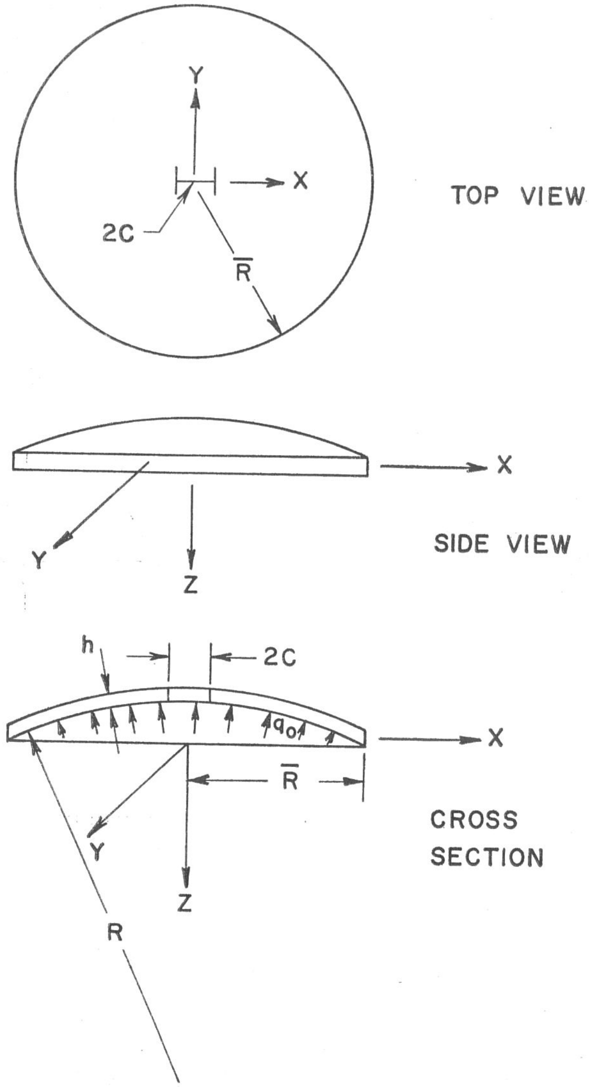

FIG. 1

fying (2.2) and (2.3), but that there is a residual normal moment M_y, equivalent vertical shear V_y, normal in-plane stress N_y, and in-plane tangential stress N_{xy}, along the real axis $|x| < 1$, of the form:

$$M_y^{(P)} = -\frac{D}{c^2} m_0, \qquad V_y^{(P)} = 0, \qquad N_y^{(P)} = -\frac{n_0}{c^2}, \qquad N_{xy}^{(P)} = 0 \qquad (24\text{-}7)$$

For simplicity, we take m_0, n_0 to be constants.*

* For m_0, n_0 non-constants, see the remarks in sect. 8.

3. Mathematical Statement of the Problem. Assuming therefore that a particular solution has been found, we need to find now two functions of the dimensionless coordinates (x, y), $W(x, y)$ and $F(x, y)$, such that they satisfy the partial differential equations (2.2) and (2.3) and the following boundary conditions. At $y = 0$ and $|x| < 1$:

$$M_y(x, 0) = -\frac{D}{c^2}\left[\frac{\partial^2 W}{\partial y^2} + \nu \frac{\partial^2 W}{\partial x^2}\right] = \frac{Dm_0}{c^2} \tag{3.1}$$

$$V_y(x, 0) = -\frac{D}{c^3}\left[\frac{\partial^3 W}{\partial y^3} + (2 - \nu)\frac{\partial^3 W}{\partial x^2 \partial y}\right] = 0 \tag{3.2}$$

$$N_y(x, 0) = \frac{1}{c^2}\frac{\partial^2 F}{\partial x^2} = \frac{n_0}{c^2}, \qquad N_{xy}(x, 0) = -\frac{1}{c^2}\frac{\partial^2 F}{\partial x \partial y} = 0 \tag{3.3, 4}$$

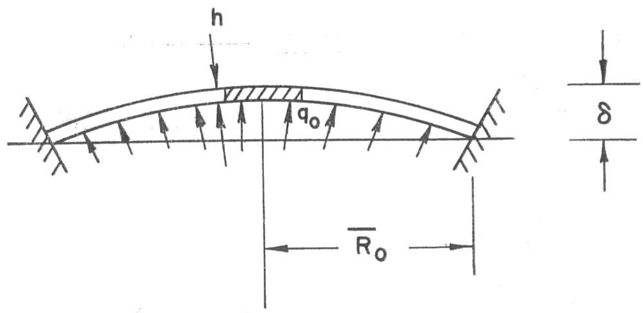

FIG. 2

At $y = 0$ and $|x| > 1$ we must satisfy the continuity requirements, i.e.

$$\lim_{|y|\to 0}\left[\frac{\partial^n}{\partial y^n}(W^+) - \frac{\partial^n}{\partial y^n}(W^-)\right] = 0 \tag{3.5}$$

$$\lim_{|y|\to 0}\left[\frac{\partial^n}{\partial y^n}(F^+) - \frac{\partial^n}{\partial y^n}(F^-)\right] = 0 \tag{3.6}$$

for $n = 0, 1, 2, 3$. Furthermore, because we are limiting ourselves to a large radius of curvature for this shallow shell, i.e., small deviations from a flat sheet, we can apply certain boundary conditions at infinity even though we know physically that the stresses and displacements far away from the crack are finite. Therefore, to avoid infinite stresses and infinite displacements we must require that the displacement function W and the stress function F with their first derivatives to be finite far away from the crack. These restrictions simplify the mathematical complexities of the problem considerably, and correspond to the usual expectations of the St. Venant Principle. It should be pointed out that the boundary conditions at infinity are not geometrically feasible. However if the crack is small compared to the dimensions of the shell, the approximation is good.

4. Reduction of the System. Reissner [3] has shown that the solution to the system (2.2), (2.3) can be written in the form

$$W = \chi + \Phi, \qquad F = -RDc^{-2}\nabla^2\chi + \psi \tag{4.1, 2}$$

where Φ and ψ are harmonic functions and χ satisfies the same differential equation as the deflection of a plate on an elastic foundation, i.e.,

$$(\nabla^4 + \lambda^4)\chi = 0 \tag{4.3}$$

where

$$\lambda^4 \equiv \frac{Ehc^4}{R^2D} \equiv \frac{12(1-\nu^2)}{(R/h)^2}\left(\frac{c}{h}\right)^4$$

The function ψ represents the inextensional bending part of the solution, and Φ represents the membrane part of the solution.

5. Integral Representations of the Solution. We next construct the following representations which have the proper symmetrical behavior with respect to x,

$$W(x, y^{\pm}) = \int_0^\infty \{P_1 \exp[-\sqrt{s^2 - i\lambda^2}\,|y|] + P_2 \exp[-\sqrt{s^2 + i\lambda^2}\,|y|] + P_3 e^{-s|y|}\} \cos xs\, ds \tag{5.1}$$

$$F(x, y^{\pm}) = \frac{i\lambda^2 RD}{c^2}\int_0^\infty \{P_1 \exp[-\sqrt{s^2 - i\lambda^2}\,|y|] - P_2 \exp[-\sqrt{s^2 + i\lambda^2}\,|y|] + P_4 e^{-s|y|}\} \cos xs\, ds \tag{5.2}$$

where the P_i are arbitrary functions of s to be determined from the boundary conditions, and the $\pm$ signs refer to $y > 0$ and $y < 0$ respectively.

Imposition of the boundary condition requirements Eq. (3.1–3.4), using Eq. (3.2) and Eq. (3.4) to determine P_3 and P_4 respectively, give for $y = 0$ and $|x| < 1$

$$-\frac{i\lambda^2 RD}{c^2}\int_0^\infty [(1 - s^{-1}\sqrt{s^2 - i\lambda^2})P_1 - (1 - s^{-1}\sqrt{s^2 + i\lambda^2})P_2]\, s^2 \cos xs\, ds = n_0 \tag{5.3}$$

$$\int_0^\infty \{[\nu_0 s^2 - i\lambda^2 - s^{-1}\sqrt{s^2 - i\lambda^2}(\nu_0 s^2 + i\lambda^2)]P_1 + [\nu_0 s^2 + i\lambda^2 - s^{-1}\sqrt{s^2 + i\lambda^2}(\nu_0^2 - i\lambda^2)]P_2\} \cos xs\, ds = -m_0 \tag{5.4}$$

whereas continuity conditions on the functions and their derivatives for $y = 0$ and $|x| > 1$ will be satisfied if

$$\int_0^\infty \begin{pmatrix} P_1\sqrt{s^2 - i\lambda^2} \\ P_2\sqrt{s^2 + i\lambda^2} \end{pmatrix} \frac{\cos xs}{s^2}\, ds = 0; \qquad |x| > | \tag{5.5, 6}$$

Therefore we have reduced our problem to solving the dual integral equations (5.3)–(5.6) for the unknown functions $P_1(s)$ and $P_2(s)$ where

$$\nu_0 s^3 P_3 = -\{\sqrt{s^2 - i\lambda^2}(\nu_0 s^2 + i\lambda^2)P_1 + \sqrt{s^2 + i\lambda^2}(\nu_0 s^2 - i\lambda^2)P_2\} \tag{5.7}$$

$$sP_4 = -(P_1\sqrt{s^2 - i\lambda^2} - P_2\sqrt{s^2 + i\lambda^2}) \tag{5.8}$$

6. Reduction to Single Integral Equations. Because we are unable to solve dual integral equations of the type discussed in the previous section, therefore we will reduce the problem to singular integral equations. Let

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \int_0^\infty \begin{pmatrix} P_1\sqrt{s^2 - i\lambda^2} \\ P_2\sqrt{s^2 + i\lambda^2} \end{pmatrix} \frac{\cos sx}{s^2}\, ds \tag{6.1, 2}$$

which by Fourier inversion gives:

$$\begin{pmatrix} P_1\sqrt{s^2 - i\lambda^2} \\ P_2\sqrt{s^2 + i\lambda^2} \end{pmatrix} = \frac{2s^2}{\pi} \int_0^1 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cos \xi s\, d\xi \tag{6.3, 4}$$

where the functions $u_1(\xi)$ and $u_2(\xi)$, due to the symmetry of the problem, are even. Formally substituting (6.3) and (6.4) into (5.3), (5.4) we find after changing the order of integration and rearranging

$$N_y = -\frac{2i\lambda^2 RD}{\pi c^2} \int_{-1}^{1} \{u_1(\xi)L_1^* - u_2(\xi)L_2^*\}\, d\xi \tag{6.5}$$

$$M_y = -\frac{2D}{\pi} \int_{-1}^{1} \{u_1(\xi)L_3^* + u_2(\xi)L_4^*\}\, d\xi \tag{6.6}$$

where

$$L_1^* \equiv \frac{1}{2}\int_0^\infty \frac{s^4 \exp[-\sqrt{s^2 - \alpha^2\lambda^2}\,|y|]}{\sqrt{s^2 - \alpha^2\lambda^2}} \cos (x - \xi)s\, ds - \frac{1}{2}\int_0^\infty s^3 e^{-s|y|} \cos (x - \xi)s\, ds \tag{6.7}$$

$$L_2^* \equiv \frac{1}{2}\int_0^\infty \frac{s^4 \exp[-\sqrt{s^2 - \beta^2\lambda^2}\,|y|]}{\sqrt{s^2 - \beta^2\lambda^2}} \cos (x - \xi)s\, ds - \frac{1}{2}\int_0^\infty s^3 e^{-s|y|} \cos (x - \xi)s\, ds \tag{6.8}$$

$$L_3^* \equiv \frac{1}{2}\int_0^\infty \left\{ \frac{s^2(\nu_0 s^2 - i\lambda^2)}{\sqrt{s^2 - \alpha^2\lambda^2}} \exp[-\sqrt{s^2 - \alpha^2\lambda^2}\,|y|] - s(\nu_0 s^2 + i\lambda^2)e^{-s|y|} \right\} \cos (x - \xi)s\, ds \tag{6.9}$$

$$L_4^* \equiv \frac{1}{2}\int_0^\infty \left\{ \frac{s^2(\nu_0 s^2 + i\lambda^2)}{\sqrt{s^2 - \beta^2\lambda^2}} \exp[-\sqrt{s^2 - \beta^2\lambda^2}\,|y|] - s(\nu_0 s^2 - i\lambda^2)e^{-s|y|} \right\} \cos (x - \xi)s\, ds \tag{6.10}$$

The integrations in (6.7)–(6.10) may be carried out explicitly by making use of the Fourier cosine transforms

$$\int_0^\infty e^{-|y|s} \cos \zeta s\, ds = \frac{|y|}{\rho^2} \tag{6.11}$$

$$\int_0^\infty \frac{\exp[-\sqrt{s^2 + a^2}\,|y|]}{\sqrt{s^2 + a^2}} \cos \zeta s\, ds = K_0(a\rho); \quad \operatorname{Re} a > 0 \tag{6.12}$$

and similar results obtained by differentiating them with respect to x and y. In these formulas $\rho^2 = \zeta^2 + |y|^2$, and K_n denotes the modified Bessel function of the third kind of order n.

The expressions (6.7)—(6.10) then become respectively

$$2L_1^* \equiv \frac{\partial}{\partial x}\left\{-\frac{\lambda^2\beta^2\zeta}{\rho^4}(\zeta^2 - 3|y|^2)K_0(\lambda\beta\rho) - \left[\frac{\lambda^3\beta^3\zeta^3}{\rho^3} + \frac{2\lambda\beta\zeta}{\rho^5}(\zeta^2 - 3|y|^2)\right]K_1(\lambda\beta\rho) + \frac{2\zeta}{\rho^4} - \frac{8\zeta|y|^2}{\rho^6}\right\} \tag{6.13}$$

$$2L_2^* \equiv \frac{\partial}{\partial x}\left\{-\frac{\lambda^2\alpha^2\zeta}{\rho^4}(\zeta^2 - 3|y|^2)K_0(\lambda\alpha\rho) - \left[\frac{\lambda^3\alpha^3\zeta^3}{\rho^3} + \frac{2\lambda\alpha\zeta}{\rho^5}(\zeta^2 - 3|y|^2)\right]K_1(\lambda\alpha\rho) + \frac{2\zeta}{\rho^4} - \frac{8\zeta|y|^2}{\rho^6}\right\} \tag{6.19}$$

$$2L_3^* \equiv \frac{\partial}{\partial x}\left\{-\frac{\nu_0\lambda^2\beta^2\zeta}{\rho^4}(\zeta^2 - 3|y|^2)K_0(\lambda\beta\rho) - \nu_0 \cdot \left[\frac{\lambda^3\beta^3\zeta^3}{\rho^3} + \frac{2\lambda\beta\zeta}{\rho^5}(\zeta^2 - 3|y|^2)\right]K_1(\lambda\beta\rho) - \frac{\alpha^2\lambda^3\beta\zeta}{\rho}K_1(\lambda\beta\rho) + \frac{2\nu_0\zeta}{\rho^4} - \frac{8\zeta|y|^2\nu_0}{\rho^6} - \frac{\alpha^2\lambda^2\zeta}{\rho^2}\right\} \tag{6.15}$$

$$2L_4^* \equiv \frac{\partial}{\partial x}\left\{-\frac{\nu_0\lambda^2\alpha^2\zeta}{\rho^4}(\zeta^2 - 3|y|^2)K_0(\lambda\alpha\rho) - \nu_0 \cdot \left[\frac{\lambda^3\alpha^3\zeta^3}{\rho^3} + \frac{2\lambda\alpha\zeta}{\rho^5}(\zeta^2 - 3|y|^2)\right]K_1(\lambda\alpha\rho) - \frac{\beta^2\lambda^3\alpha\zeta}{\rho}K_1(\lambda\alpha\rho) + \frac{2\nu_0\zeta}{\rho^4} - \frac{8\zeta|y|^2\nu_0}{\rho^6} - \frac{\beta^2\lambda^2\zeta}{\rho^2}\right\} \tag{6.16}$$

Thus, the limits as $|y| \to 0$ of N_y and M_y are found to be respectively

$$\begin{aligned} \lim_{|y|\to 0} N_y &= -\frac{2i\lambda^2 RD}{\pi c^2}\frac{d}{dx}\int_{-1}^{1}\{u_1(\xi)L_1 - u_2(\xi)L_2\}\,d\xi \\ \lim_{|y|\to 0} M_y &= -\frac{2D}{\pi}\frac{d}{dx}\int_{-1}^{1}\{u_1(\xi)L_3 + u_2(\xi)L_4\}\,d\xi \end{aligned} \tag{6.17}$$

where the integrals are understood to be of Cauchy principal value and

$$2L_1 \equiv -\frac{\lambda^2\beta^2}{\zeta}K_0(\lambda\beta|\zeta|) - \left(\lambda^3\beta^3\frac{\zeta}{|\zeta|} + \frac{2\lambda\beta}{\zeta|\zeta|}\right)K_1(\lambda\beta|\zeta|) + \frac{2}{\zeta^3} \tag{6.18}$$

$$2L_2 \equiv -\frac{\lambda^2\alpha^2}{\zeta}K_0(\lambda\alpha|\zeta|) - \left(\lambda^3\alpha^3\frac{\zeta}{|\zeta|} + \frac{2\lambda\alpha}{\zeta|\zeta|}\right)K_1(\lambda\alpha|\zeta|) + \frac{2}{\zeta^3} \tag{6.19}$$

$$2L_3 \equiv -\frac{\nu_0\lambda^2\beta^2}{\zeta}K_0(\lambda\beta|\zeta|) - \nu_0\left(\lambda^3\beta^3\frac{\zeta}{|\zeta|} + \frac{2\lambda\beta}{\zeta|\zeta|}\right)K_1(\lambda\beta|\zeta|) - \alpha^2\lambda^3\beta\frac{\zeta}{|\zeta|}K_1(\lambda\beta|\zeta|) + \frac{2\nu_0}{\zeta^3} - \frac{\alpha^2\lambda^2}{\zeta} \tag{6.20}$$

$$2L_4 \equiv -\frac{\nu_0\lambda^2\alpha^2}{\zeta} K_0(\lambda\alpha \mid \zeta \mid) - \nu_0\left(\lambda^3\alpha^3 \frac{\zeta}{\mid \zeta \mid} + \frac{2\lambda\alpha}{\zeta \mid \zeta \mid}\right) K_1(\lambda\alpha \mid \zeta \mid) \\ -\beta^2\lambda^3\alpha \frac{\zeta}{\mid \zeta \mid} K_1(\lambda\alpha \mid \zeta \mid) + \frac{2\nu_0}{\zeta^3} - \frac{\beta^2\lambda^2}{\zeta} \tag{6.21}$$

If we set N_y, M_y, in the limit as $\mid y \mid \to 0$, equal to $-n_0$ and $-m_0$ respectively, integrate with respect to x, then we find that they must satisfy the integral equations

$$\int_{-1}^{1} \{u_1(\xi)2L_1 - u_2(\xi)2L_2\}\, d\xi = -\frac{\pi n_0 c^2}{i\lambda^2 RD} x; \qquad \mid x \mid < 1 \tag{6.22}$$

$$\int_{-1}^{1} \{u_1(\xi)2L_3 + u_2(\xi)2L_4\}\, d\xi = -\pi m_0 x; \qquad \mid x \mid < 1 \tag{6.23}$$

The kernels L_1, L_2, L_3, L_4 have singularities of the order $1/\zeta \equiv 1/(x - \xi)$, as can easily be seen by observing their behavior for small arguments:

$$2L_1 = -\frac{\lambda^2\beta^2}{2(x - \xi)} + \lambda^4\beta^4(x - \xi)\left[\frac{5}{32} - \frac{3\gamma}{8} - \frac{3}{8}\ln \lambda\beta \frac{\mid x - \xi \mid}{2}\right] \\ + O(\lambda^6(x - \xi)^3 \ln \lambda \mid x - \xi \mid) \tag{6.24}$$

$$2L_2 = -\frac{\lambda^2\alpha^2}{2(x - \xi)} + \lambda^4\alpha^4(x - \xi)\left[\frac{5}{32} - \frac{3\gamma}{8} - \frac{3}{8}\ln \lambda\alpha \frac{\mid x - \xi \mid}{2}\right] \\ + O(\lambda^6(x - \xi)^3 \ln \lambda \mid x - \xi \mid) \tag{6.25}$$

$$2L_3 = -\frac{\lambda^2\alpha^2(4 - \nu_0)}{2(x - \xi)} + \lambda^4\beta^4(x - \xi)\left[\frac{5\nu_0 - 8}{32} + \frac{4 - 3\nu}{8} \right. \\ \left. \cdot \left(\gamma + \ln \lambda\beta \frac{\mid x - \xi \mid}{2}\right)\right] + O(\lambda^6(x - \xi)^3 \ln \lambda \mid x - \xi \mid) \tag{6.26}$$

$$2L_4 = -\frac{\lambda^2\beta^2(4 - \nu_0)}{2(x - \xi)} + \lambda^4\alpha^4(x - \xi)\left[\frac{5\nu_0 - 8}{32} + \frac{4 - 3\nu_0}{8} \right. \\ \left. \cdot \left(\gamma + \ln \frac{\lambda \mid x - \xi \mid}{2}\right)\right] + O(\lambda^6(x - \xi)^3 \ln \lambda \mid x - \xi \mid) \tag{6.27}$$

We require that the solutions $u_1(x)$, $u_2(x)$ be Hölder continuous for some positive Hölder indices μ_1 and μ_2 for all x in the closed interval $[-1, 1]$. Thus in particular $u_1(x)$, $u_2(x)$ are to be bounded near the ends of the crack.

The problem of obtaining a solution to the coupled integral equations (6.22) and (6.23) can be reduced to the problem of solving two coupled Fredholm integral equations with a bounded kernel. See Reference 4.

7. Approximate Integral Equations. Because of the complicated nature of the kernels L_1, L_2, L_3 and L_4, an exact solution for the unknown functions $u_1(x)$ and $u_2(x)$ is extremely difficult. On the other hand, for most practical applications the parameter λ attains small values as follows from the definition of λ namely

$$\lambda \equiv \frac{\sqrt[4]{12(1-\nu^2)}}{\sqrt{R/h}}(c/h) = \sqrt[4]{12(1-\nu^2)}\,(c/R)(R/h)^{\frac{1}{2}}$$

It is clear that λ is small for large ratios of R/h and small crack lengths. As a practical matter, if we consider crack lengths less than one tenth of the periphery, i.e. $2c < 2\pi R/10$, and for $R/h < 10^3$ a corresponding upper bound for λ can be obtained, namely $\lambda < 20$.* Thus the range of λ becomes $0 < \lambda < 20$ and for most practical cases is between 0 and 2, depending upon the size of the crack.

If we consider small λ, we may replace the exact singular integral equations with the following approximate set

$$\int_{-1}^{1} \{u_{10}(\xi)2l_1 - u_{20}(\xi)2l_2\}\, d\xi = -\frac{n_0 c^2 \pi}{i\lambda^2 RD} x; \qquad |x| < 1 \quad (7.1)$$

$$\int_{-1}^{1} u_{10}(\xi)2l_3 + u_{20}(\xi)2l_4\, d\xi = -m_0 \pi x; \qquad |x| < 1 \quad (7.2)$$

where the new kernels are:

$$2l_1 \equiv \frac{\lambda^2\beta^2}{2(x-\xi)} + \lambda^4\beta^4(x-\xi)\left[\frac{5}{32} - \frac{3\gamma}{8} - \frac{3}{8}\ln\frac{\lambda\beta\,|x-\xi|}{2}\right] \quad (7.3)$$

$$2l_2 \equiv -\frac{\lambda^2\alpha^2}{2(x-\xi)} + \lambda^4\alpha^4(x-\xi)\left[\frac{5}{32} - \frac{3\gamma}{8} - \frac{3}{8}\ln\frac{\lambda\alpha\,|x-\xi|}{2}\right] \quad (7.4)$$

$$2l_3 \equiv -\frac{\alpha^2\lambda^2(4-\nu_0)}{2(x-\xi)} + \lambda^4\beta^4(x-\xi)\left[\frac{5\nu_0 - 8}{32} + \frac{4-3\nu_0}{8} \cdot \left(\gamma + \ln\frac{\lambda\beta\,|x-\xi|}{2}\right)\right] \quad (7.5)$$

$$2l_4 \equiv -\frac{\beta^2\lambda^2(4-\nu_0)}{2(x-\xi)} + \lambda^4\alpha^4(x-\xi)\left[\frac{5\nu_0 - 8}{32} + \frac{4-3\nu_0}{8} \cdot \left(\gamma + \ln\frac{\lambda\alpha\,|x-\xi|}{2}\right)\right] \quad (7.6)$$

8. Solutions to Approximate Integral Equations for Small λ. For the simple case $\lambda = 0$ the problem reduces to that of a flat sheet under applied bending and stretching loads, the solution of which has been investigated by many authors. For example, the problem for both bending and stretching for an orthotropic plate, containing a finite crack, was investigated by Ang and Williams [1] and a solution was obtained by means of dual integral equations. It can easily be shown† that the dual integral equations can be transformed to two singular

* Dr. E. Reissner has suggested that the physical situation corresponding to $\lambda \gg 1$ might be susceptible to treatment as a boundary layer problem.

† See Noble [5].

integral equations of the type (6.22) and (6.23) with simpler kernels. Furthermore, these are not coupled and the solutions can easily be obtained as in §47 of [6]. Without going into the details they are found to be of the form $A\sqrt{1-\xi^2}$, where A is a constant.

Similarly, the solution for an initially curved sheet must, in the limit, check the above result and because $u_1(\xi)$ and $u_2(\xi)$ are in particular to be bounded near the ends of the crack, it is reasonable to assume solutions of the form

$$u_{10}(\xi) = \sqrt{1-\xi^2}[A_1 + \lambda^2 A_2(1-\xi^2) + \cdots]; \qquad |\xi| < 1 \quad (8.1)$$

$$u_{20}(\xi) = \sqrt{1-\xi^2}[B_1 + \lambda^2 B_2(1-\xi^2) + \cdots]; \qquad |\xi| < 1 \quad (8.2)$$

where the coefficients $A_1, A_2, \cdots, B_1, B_2, \cdots$ can be functions of λ but not of ξ.

Substituting (8.1) and (8.2) into (7.1) and (7.2) and making use of the relation

$$\int_{-1}^{1} \sqrt{1-\xi^2}\,(x-\xi) \ln \frac{\lambda\alpha\,|x-\xi|}{2}\, d\xi = \frac{\pi}{4}\left(1 + \ln \frac{\lambda^2\alpha^2}{16}\right) x + \frac{\pi}{6} x^3$$

we equate coefficients and obtain

$$\begin{aligned} A_1 = \frac{n_0 c^2}{\lambda^4 RD} \Bigg\{ 1 &+ \frac{\pi\lambda^2}{16}\frac{8-3\nu_0}{4-\nu_0} + \frac{\lambda^2\alpha^2}{4-\nu_0}\left(\frac{8-7\nu_0}{32} + \frac{4-3\nu_0}{8}\gamma\right) \\ &+ \frac{\lambda^2\alpha^2}{16}\frac{4-3\nu_0}{4-\nu_0}\left(1+\ln\frac{\lambda^2\alpha^2}{16}\right)\Bigg\} + \frac{m_0}{\lambda^2\alpha^2(4-\nu_0)}\Bigg\{1 + \frac{\pi\lambda^2}{16}\frac{8-3\nu_0}{4-\nu_0} \\ &+ \lambda^2\alpha^2\left(\frac{7}{32}+\frac{3\gamma}{8}\right) + \frac{3}{16}\lambda^2\alpha^2\left(1+\ln\frac{\lambda^2\alpha^2}{16}\right)\Bigg\} + O(\lambda^2\ln\lambda) \end{aligned} \quad (8.3)$$

$$\begin{aligned} B_1 = \frac{n_0 c^2}{\lambda^4 RD} \Bigg\{ 1 &+ \frac{\lambda^2\pi}{16}\frac{8-3\nu_0}{4-\nu_0} + \frac{\lambda^2\beta^2}{4-\nu_0}\left(\frac{8-7\nu_0}{32} + \frac{4-3\nu_0}{8}\gamma\right) \\ &+ \frac{\lambda^2\beta^2}{16}\frac{4-3\nu_0}{4-\nu_0}\left(1+\ln\frac{\lambda^2\beta^2}{16}\right)\Bigg\} + \frac{m_0}{\lambda^2\beta^2(4-\nu_0)}\Bigg\{1 + \frac{\lambda^2\pi}{16}\frac{8-3\nu_0}{4-\nu_0} \\ &+ \lambda^2\beta^2\left(\frac{7}{32}+\frac{3\gamma}{8}\right) + \frac{3}{16}\lambda^2\beta^2\left(1+\ln\frac{\lambda^2\beta^2}{16}\right)\Bigg\} + O(\lambda^2\ln\lambda) \end{aligned} \quad (8.4)$$

We should point out that, if coefficients A_1, B_1 of higher accuracy are desired, say up to order λ^{2n}, then it is necessary to solve an $n \times n$ algebraic system. In effect, this is a method of successive approximations for which the question of convergence is investigated in Reference 4.

It thus appears that for $\lambda < \lambda^*$ the power series solutions of the form

$$u_1^{(N)}(\xi) = \sqrt{1-\xi^2}\,\textstyle\sum_{n=0}^{N} A_{n+1}\lambda^{2n}(1-\xi^2)^n, \quad (8.5)$$

$$u_2^{(N)}(\xi) = \sqrt{1-\xi^2}\,\textstyle\sum_{n=0}^{N} B_{n+1}\lambda^{2n}(1-\xi^2)^n \quad (8.6)$$

in the limit as $N \to \infty$, will converge to the exact solutions $u_1(\xi)$ and $u_2(\xi)$ of the integral equations (7.1) and (7.2). However since most particular solutions will give us a non-uniform residual moment and normal membrane stress along the crack, it is only natural to ask how the solution changes. Suppose, for $|x| < 1$, we expand m_0 and n_0 in the form $\sum_n a_n x^{2n}$ (even powers because of the symmetry of the problem), then our previous method of solution will still be applicable. And as can easily be seen from equations (6.22) and (6.23), although the coefficients A_n, B_n in this case may change, the character of the solution will still remain the same. Finally, because we desire to focus our attention upon the singular stresses around the neighborhood of the crack point, we need only to compute coefficients A_1 and B_1.

9. Determination of W and F. In view of equations (6.4), (6.5), (8.1), (8.2) and the relation

$$\int_0^\infty s^{-\mu} J_\mu(as) \cos xs \, ds$$
$$= \begin{cases} \sqrt{\pi}\,(2a)^{-\mu}[\Gamma(\mu + \tfrac{1}{2})]^{-1}(a^2 - x^2)^{\mu-\frac{1}{2}}; & 0 < x < a; \quad \operatorname{Re}\mu > -\tfrac{1}{2} \\ 0; & a < x < \infty; \quad \operatorname{Re}\mu > -\tfrac{1}{2} \end{cases} \tag{9.1}$$

which can be found on page 44 of [7] we have:

$$P_1(s) = \frac{s}{\sqrt{s^2 - i\lambda^2}}\left\{A_1 J_1(s) + 3\lambda^2 A_2 \frac{J_2(s)}{s} + O(\lambda^4)\right\} \tag{9.2}$$

and similarly

$$P_2(s) = \frac{s}{\sqrt{s^2 + i\lambda^2}}\left\{B_1 J_1(s) + 3\lambda^2 B_2 \frac{J_2(s)}{s} + O(\lambda^4)\right\} \tag{9.3}$$

where A_1 and B_1 are given by (8.3) and (8.4) respectively. And finally substituting (9.2) and (9.3) into (5.7) and (5.8) we find

$$\begin{aligned} P_3(s) = &-(A_1 + B_1)J_1(s) - 3\lambda^2(A_2 + B_2)\frac{J_2(s)}{s} \\ &- \frac{i\lambda^2}{\nu_0 s^2}(A_1 - B_1)J_1(s) + O(\lambda^4) \end{aligned} \tag{9.4}$$

and

$$P_4(s) = -(A_1 - B_1)J_1(s) - 3\lambda^2(A_2 - B_2)s^{-1}J_2(s) + O(\lambda^4) \tag{9.5}$$

Therefore a substitution of the above relations into (5.1) and (5.2) will determine the bending deflection W and membrane stress function F. It is clear that the integrals in equations (5.1) and (5.2) converge and the differentiations under the integral signs are also justified at least for $y \neq 0$. The values of the derivatives at $y = 0$, $x < 0$ can be obtained by a proper limiting process.

10. Determination of the Singular Stresses. In view of equations (5.1), (5.2), (9.2), (9.3), the bending and extensional stresses defined by

$$\sigma_{x_b} = -\frac{EZ}{(1 - \nu^2)c^2}\left[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right] \tag{10.1}$$

$$\sigma_{y_b} = -\frac{EZ}{(1-\nu^2)c^2}\left[\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right] \tag{10.2}$$

$$\tau_{xy_b} = -\frac{2GZ}{c^2}\frac{\partial^2 w}{\partial x \partial y} \tag{10.3}$$

$$\sigma_{x_e} = \frac{1}{hc^2}\frac{\partial^2 F}{\partial y^2}, \qquad \sigma_{y_e} = \frac{1}{hc^2}\frac{\partial^2 F}{\partial x^2}, \qquad \tau_{xy_e} = -\frac{1}{hc^2}\frac{\partial^2 F}{\partial x \partial y} \tag{10.4–6}$$

can be computed. Without going into the details we list below the results.

*Bending Stresses:** On the surface $Z = +\frac{1}{2}h$

$$\sigma_{x_b} = -\frac{Eh}{2(1+\nu)c^2}\frac{\tilde{P}_{10}}{\sqrt{\epsilon}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{5\theta}{2}\right) + O(\epsilon^0) \tag{10.7}$$

$$\sigma_{y_b} = \frac{Eh}{2(1-\nu^2)c^2}\frac{\tilde{P}_{10}}{\sqrt{\epsilon}}\left(\frac{11+5\nu}{4}\cos\frac{\theta}{2} + \frac{1-\nu}{4}\cos\frac{5\theta}{2}\right) + O(\epsilon^0) \tag{10.8}$$

$$\tau_{xy_b} = -\frac{Gh}{(1-\nu)c^2}\frac{\tilde{P}_{10}}{\sqrt{\epsilon}}\left(\frac{7+\nu}{4}\sin\frac{\theta}{2} + \frac{1-\nu}{4}\sin\frac{5\theta}{2}\right) + O(\epsilon^0) \tag{10.9}$$

where

$$\begin{aligned}\tilde{P}_{10} \equiv \alpha^2\lambda^2\frac{A_1 - B_1}{2\sqrt{2}} = &-\frac{n_0\lambda^2}{\sqrt{2EhD}\,(4-\nu_0)}\left\{\frac{8-7\nu_0}{32} + \frac{4-3\nu_0}{8}\gamma\right. \\ &\left. + \frac{4-3\nu_0}{16}\left(1 + \ln\frac{\lambda^2}{16}\right)\right\} \\ &+ \frac{m_0}{\sqrt{2}\,(4-\nu_0)}\left\{1 + \frac{\pi\lambda^2}{32}\frac{4-3\nu_0}{4-\nu_0}\right\} + O(\lambda^4\ln\lambda)\end{aligned} \tag{10.10}$$

Similarly

Extensional stresses:

$$\sigma_{x_e} = \frac{\tilde{P}_{20}}{hc^4\sqrt{\epsilon}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{5\theta}{2}\right) + O(\epsilon^0) \tag{10.11}$$

$$\sigma_{y_e} = \frac{\tilde{P}_{20}}{hc^4\sqrt{\epsilon}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{5\theta}{2}\right) + O(\epsilon^0) \tag{10.12}$$

$$\tau_{xy_e} = -\frac{\tilde{P}_{20}}{hc^4\sqrt{\epsilon}}\left(\frac{1}{4}\sin\frac{\theta}{2} - \frac{1}{4}\sin\frac{5\theta}{2}\right) + O(\epsilon^0) \tag{10.13}$$

where

$$\begin{aligned}\tilde{P}_{20} &\equiv \frac{\lambda^4 RD}{2\sqrt{2}}(A_1 + B_1) \\ &= \frac{n_0 c^2}{\sqrt{2}}\left\{1 + \frac{3\pi}{32}\lambda^2\right\} + \frac{m_0\lambda^2\sqrt{EhD}\,c^2}{\sqrt{2}\,(4-\nu_0)}\left\{\frac{13}{32} + \frac{3\gamma}{8} + \frac{3}{16}\ln\frac{\lambda^2}{16}\right\} \\ &\quad + O(\lambda^4\ln\lambda)\end{aligned} \tag{10.14}$$

* Note because of the Kirchhoff boundary conditions, the bending shear stress does not vanish in the free edge. For the flat sheet this problem was discussed by Knowles and Wang [8].

It is apparent from the above equations that there exists an interaction between bending and stretching, except that in the limit as $\lambda \to 0$ the stresses of a flat sheet are recovered and coincide with those obtained previously for bending [9] and extension [10]. Thus the stresses in a shell are expressed in terms of the stresses in a flat sheet.

11. Combined Stresses. In general, the combined stresses will depend upon the contributions of the particular solutions reflecting the magnitude and distribution of the applied normal pressure. On the other hand the singular part of the solution, that is the terms producing infinite elastic stresses at the crack tip, will depend only upon the local stresses existing along the locus of the crack before it is cut, which of course are precisely the stresses which must be removed or cancelled by the particular solutions described above in order to obtain the stress free edges as required physically. Hence the distribution of $q(x, y)$ does not—to the first order—affect the local character of the stresses at the crack point.

It is believed of more than passing interest to observe that for practical purposes, the final representation of the stress field in a spherically curved shallow shell can be viewed as the flat plate distribution modified by a correction factor, essentially of the form $1 + O(\lambda^2)$. To the extent that elastic analysis can anticipate fracture in the presence of ductility produced by the mathematically infinite stresses near the crack tip, one would conjecture that the critical applied stress at fracture might also be expressible in essentially the same form, i.e. reduced by a factor $1 - O(\lambda^2)$. It may also be noted that if such a correlation could be established, a considerable amount of difficult curved panel testing could be eliminated, or replaced by the simpler flat plate geometry.

By way of illustration of how this solution can be applied in a specific situation, the special case of a segment of a sphere subjected to uniform loading has been investigated by the author and published elsewhere [11].

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