Math 5440 Problem Set 4 – Solutions

1: (Logan, 1.8 # 4) Find all radial solutions of the two-dimensional Laplace’s equation. That is, find all solutions of the form $u(r)$ where $r = \sqrt{x^2 + y^2}$. Find the steady-state temperature distribution in the annular domain $1 \leq r \leq 2$ if the inner circle $r = 1$ is held at 0 degrees and the outer circle $r = 2$ is held at 10 degrees.

We seek $u(r)$ that satisfies

$$0 = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} = \frac{1}{r} \left\{ r \frac{\partial u}{\partial r} \right\}.$$ 

Integrating, we see that $ru_r = C$ for some constant $C$ and integrating again we obtain $u(r) = C \ln(r) + B$ where $B$ and $C$ are constants. To satisfy the given conditions, we need $0 = u(1) = B$, so $B = 0$, and $10 = u(2) = C \ln(2)$, so $C = 10 / \ln(2)$. Hence

$$u(r) = 10 \frac{\ln r}{\ln 2}.$$ 

$\blacksquare$
2: (Logan, 1.9 # 1) Classify the PDE
\[ u_{xx} + 2k u_{xt} + k^2 u_{tt} = 0, \quad k \neq 0. \]

Find a transformation \( \xi = x + bt, \tau = x + dt \) of the independent variables that transforms the equation into a simpler equation of the form \( U_{\xi \xi} = 0 \). Find the solution to the given equation in terms of two arbitrary functions.

The equation
\[ u_{xx} + 2k u_{xt} + k^2 u_{tt} = 0 \quad k \neq 0, \]
has \( A = 1, B = 2k, \) and \( C = k^2 \), so \( D = 4k^2 - 4k^2 = 0 \) and the equation is therefore parabolic. Letting \( \xi = x + bt \) and \( \tau = x + dt \) converts the equation to
\[
(1 + 2kb + k^2b^2)U_{\xi \xi} + (2 + 2k(d + b) + 2k^2bd)U_{\xi \tau} + (1 + 2kd + k^2d^2)U_{\tau \tau} = 0
\]
and choosing \( d = -1/k \) and \( b = 0 \) this becomes
\[ U_{\xi \xi} = 0. \]

Integrating twice, we find that
\[ U(\xi, \tau) = F(\tau)\xi + G(\tau) \]
for arbitrary functions \( F \) and \( G \). In terms of \( x \) and \( t \), the solution is
\[ u(x, t) = F(x - t/k)x + G(x - t/k). \]
3: (Logan, 1.9 # 4) Show that the equation
\[ u_{tt} - c^2 u_{xx} + au_t + bu_x + du = f(x,t) \]
can be transformed into the form
\[ w_{\xi \tau} + kw = g(\xi, \tau), \quad w = w(\xi, \tau), \]
by first making the transformation \( \xi = x - ct, \tau = x + ct \), and then letting
\( u = w \exp(\alpha \xi + \beta \tau) \) for some choice of \( \alpha, \beta \).

Let \( \xi = x - ct \) and \( \tau = x + ct \), \( u(x,t) = U(\xi, \tau) = U(x - ct, x + ct) \), and \( f(x,t) = F(\xi, \tau) \).
Then, we see that
\[ u_{xx} = U_{\xi \xi} + 2U_{\xi \tau} + U_{\tau \tau}, \]
and
\[ u_{tt} = U_{\xi \xi} c^2 - 2c^2 U_{\xi \tau} + c^2 U_{\tau \tau}. \]
Hence the PDE \( f = u_{tt} - c^2 u_{xx} + au_t + bu_x + du \), becomes
\[ F = -4c^2 U_{\xi \tau} + (b - ac)U_{\xi} + (b + ac)U_{\tau} + dU. \]

Letting \( U(\xi, \tau) = w(\xi, \tau) \exp(\alpha \xi + \beta \tau) \), note that \( U_{\xi} = w_{\xi} \exp(\alpha \xi + \beta \tau) + \alpha w \exp(\alpha \xi + \beta \tau) \), \( U_{\tau} = w_{\tau} \exp(\alpha \xi + \beta \tau) + \beta w \exp(\alpha \xi + \beta \tau) \), and \( U_{\xi \tau} = (w_{\tau \tau} + \beta w_{\tau} + \alpha w_{\tau} + \alpha \beta w) \exp(\alpha \xi + \beta \tau) \).
Substituting these into the PDE we get
\[ F = \left[ -4c^2 \{ w_{\xi \tau} + \beta w_{\tau} + \alpha w_{\tau} + \alpha \beta w \} + (b - ac)(w_{\xi} + \alpha w) \right. \]
\[ + \left. (b + ac)(w_{\tau} + \beta w) + dw \exp(\alpha \xi + \beta \tau) \right], \]
or
\[ F = \left[ -4c^2 w_{\xi \tau} + (b - ac - 4\beta c^2)w_{\xi} + (b + ac - 4\alpha c^2)w_{\tau} + (-4c^2 \alpha \beta + \alpha(b - ac) + \beta(b + ac) + d)w \right] \times \exp(\alpha \xi + \beta \tau) \]
Setting \( \beta = (b - ac) / 4c^2 \) and \( \alpha = (b + ac) / 4c^2 \) kills off the first derivative terms and gives
us an equation of the desired form. \[ \blacksquare \]
4: (Logan, 2.1 # 1) Solve the Cauchy problem

\[ u_t = ku_{xx}, \quad -\infty < x < \infty \text{ and } t > 0, \]

\[ u(x,0) = \phi(x), \quad -\infty < x < \infty, \]

for each of the following initial conditions

a) \( \phi(x) = 1 \) if \(|x| < 1\) and \( \phi(x) = 0 \) if \(|x| > 1\).

b) \( \phi(x) = \exp(-x) \) if \( x > 0 \) and \( \phi(x) = 0 \) if \( x < 0 \).

For each part, set \( k = 0.001 \) and use Maple to plot the solution as a function of \( x \) for \( t = 0, 0.5, 1.0, 1.5 \).

a) For

\[ \phi(x) = \begin{cases} 1 & |x| < 1, \\ 0 & |x| > 1, \end{cases} \]

\[ u(x,t) = \int_{-\infty}^{\infty} G(x-y,t) \phi(y) dy = \int_{-1}^{1} e^{-\frac{(x-y)^2}{4\beta t}} d\gamma. \]

Letting \( r = \frac{y-x}{\sqrt{4\beta t}} \), this formula becomes

\[ u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{r^2}{4\beta t}} dr \]

\[ = \frac{1}{\sqrt{\pi}} \int_{-\sqrt{\frac{1-x}{4\beta t}}}^{\sqrt{\frac{1-x}{4\beta t}}} e^{-r^2} dr - \frac{1}{\sqrt{\pi}} \int_{-\sqrt{\frac{(1+x)}{4\beta t}}}^{\sqrt{\frac{(1+x)}{4\beta t}}} e^{-r^2} dr \]

\[ = \frac{1}{2} \text{erf} \left( \frac{1-x}{\sqrt{4\beta t}} \right) - \frac{1}{2} \text{erf} \left( \frac{-1-x}{\sqrt{4\beta t}} \right). \]

b) For

\[ \phi(x) = \begin{cases} e^{-x} & x > 0, \\ 0 & x < 0, \end{cases} \]

\[ u(x,t) = \int_{-\infty}^{\infty} G(x-y,t) \phi(y) dy = \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4\beta t}} e^{-y} dy \]

\[ = \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4\beta t}} \frac{e^{-y}}{\sqrt{4\pi\beta t}} dy \]

\[ = \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4\beta t}-y} \frac{e^{-\frac{y}{4\beta t}}}{\sqrt{4\pi\beta t}} dy. \]
We complete the square in the exponent

\[- \frac{y^2 - 2xy + x^2}{4\beta t} - y = - \left\{ \frac{y^2 - (4\beta t - 2x)y + x^2}{4\beta t} \right\} = - \left\{ \frac{(y + 2\beta t - x)^2 + 4\beta tx - 4\beta^2 t^2}{4\beta t} \right\} = - \frac{(y + 2\beta t - x)^2}{4\beta t} - x + \beta t.\]

Substituting this into the formula above for \( u(x, t) \) we get

\[ u(x, t) = e^{\beta t - x} \frac{1}{\sqrt{4\pi \beta t}} \int_0^\infty e^{-\frac{(y + 2\beta t - x)^2}{4\beta t}} dy. \]

Letting \( s = -\frac{(y + 2\beta t - x)}{\sqrt{4\beta t}} \), we get

\[ u(x, t) = e^{\beta t - x} \frac{1}{\sqrt{\pi}} \int_{\frac{2\beta t - s}{\sqrt{4\beta t}}}^\infty e^{-s^2} ds = \frac{1}{2} e^{\beta t - x} \left\{ 1 - \text{erf} \left( \frac{2\beta t - x}{\sqrt{4\beta t}} \right) \right\}. \]
Figure 0.2: Plots of solution at times 0, 0.5, 1, 1.5, and 2.0.
If $|\phi(x)| \leq M$ for all $x$, where $M$ is a positive constant, show that the solution $u$ to the Cauchy problem

$$u_t = ku_{xx}, \quad -\infty < x < \infty \text{ and } t > 0,$$

$$u(x,0) = \phi(x), \quad -\infty < x < \infty,$$

satisfies $|u(x,t)| \leq M$ for all $x$ and $t > 0$. Hint: Use the calculus fact that the absolute value of an integral is less than or equal to the integral of the absolute value.

The solution to the IVP $u_t = \beta u_{xx}$ for $-\infty < x < \infty$ and $t > 0$, with $u(x,0) = \phi(x)$ for $-\infty < x < \infty$ is

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\phi(y)dy.$$ 

If $\phi(x) \leq M$ for all $x$, then

$$|u(x,t)| = \left| \int_{-\infty}^{\infty} G(x-y,t)\phi(y)dy \right|$$

$$\leq \int_{-\infty}^{\infty} |G(x-y,t)\phi(y)|dy$$

$$= \int_{-\infty}^{\infty} G(x-y,t)|\phi(y)|dy$$

$$\leq \int_{-\infty}^{\infty} G(x-y,t)Mdy$$

$$= M.$$

The last step uses the fact that $\int_{-\infty}^{\infty} G(x-y,t)dy = 1$ for all $x$ and any $t > 0$. 

$\blacksquare$
Show that if \( u(x,t) \) and \( v(x,t) \) are any two solutions to the one-dimensional heat equation \( u_t = \beta u_{xx} \), then \( w(x,y,t) = u(x,t)v(y,t) \) solves the two-dimensional heat equation, \( w_t = \beta (w_{xx} + w_{yy}) \). Guess the solution to the two-dimensional Cauchy problem

\[
w_t = \beta (w_{xx} + w_{yy}), \quad -\infty < x < \infty, -\infty < y < \infty, t > 0,
\]
\[
w(x,y,0) = \psi(x,y), \quad -\infty < x < \infty, -\infty < y < \infty.
\]

Suppose that \( u(x,t) \) and \( v(x,t) \) solve the one-dimensional diffusion problem \( u_t = \beta u_{xx} \). Let \( w(x,y,t) = u(x,t)v(y,t) \). Then \( w_t = uv_t + u_t v = u \beta v_{yy} + \beta u_{xx} v = \beta (u v_{yy} + (uv)_{yy}) = \beta (w_{xx} + w_{yy}) \). We have therefore shown that \( w(x,y,t) \) satisfies the two-dimensional diffusion equation. If we let \( u(x,t) = G(x,t) \) and \( v(y,t) = G(y,t) \), then \( w(x,y,t) = G(x,t)G(y,t) \) solves the two-dimensional diffusion equation and \( w(x,y,0) = \delta(x)\delta(y) \), so this function is the fundamental solution of the two-dimensional diffusion equation. The solution to the two-dimensional initial value problem \( w_t = \beta (w_{xx} + w_{yy}) \) for \(-\infty < x < \infty, -\infty < y < \infty, \) and \( t > 0 \), with \( w(x,y,0) = \psi(x,y) \) is

\[
w(x,y,t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x-x',t)G(y-y',t)\psi(x',y')dx'dy'.
\]
Suppose \( u_{tt} = c^2 u_{xx} \) for \(-\infty < x < \infty\) and \( t > 0 \), and \( u_t(x,0) = g(x) = 0 \) and 
\[
    u(x,0) = f(x) = \begin{cases} 
    1 - |x|/2 & \text{for } |x| < 2 \\
    0 & \text{for } |x| \geq 2.
\end{cases}
\]
Let \( c = 1 \) and find the solution to this problem. Use Maple to plot the solution at \( t=0,1,2,3,4,5,6 \).

Since \( g(x) = 0 \) for all \( x \), the solution to this problem is 
\[
    u(x,t) = \frac{1}{2} \left( f(x-ct) + f(x+ct) \right).
\]
There are various cases depending on whether \( |x - ct| < 2 \) and whether \( |x + ct| < 2 \).

\[
    u(x,t) = \begin{cases} 
    \frac{1}{2} \left\{ 1 - \frac{|x+ct|}{2} + 1 - \frac{|x-ct|}{2} \right\}, & \text{if } |x + ct| < 2 \text{ and } |x - ct| < 2 \\
    \frac{1}{2} \left\{ 1 - \frac{|x+ct|}{2} \right\}, & \text{if } |x + ct| < 2 \text{ and } |x - ct| \geq 2 \\
    \frac{1}{2} \left\{ 1 - \frac{|x-ct|}{2} \right\}, & \text{if } |x + ct| \geq 2 \text{ and } |x - ct| < 2 \\
    0, & \text{if } |x + ct| \geq 2 \text{ and } |x - ct| \geq 2.
\end{cases}
\]

Figure 0.3: Plot in \( xt \)-plane. In region A, \( |x - ct| < 2 \) and \( |x + ct| < 2 \); in region B, \( |x - ct| > 2 \) and \( |x + ct| < 2 \); in region C, \( |x - ct| < 2 \) and \( |x + ct| > 2 \); in regions D,E, and F, \( |x - ct| > 2 \) and \( |x + ct| > 2 \).

Figure 0.4: Plots of solution at times 0, 1, 2, 3, 4, 5, and 6.
8: Suppose \( u_{tt} = c^2 u_{xx} \) for \( -\infty < x < \infty \) and \( t > 0 \), and \( u(x,0) = f(x) = 0 \) and \( u_t(x,0) = g(x) = \begin{cases} 1 & \text{for } |x| < 2 \\ 0 & \text{for } |x| \geq 2. \end{cases} \)

Let \( c = 1 \) and find the solution to this problem. Use Maple to plot the solution at \( t=0,1,2,3,4,5,6. \)

Since \( f(x) = 0 \), the D’Alembert solution to the wave equation is

\[
u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, ds.
\]

For the given initial velocity, we obtain

\[
u(x,t) = \begin{cases} 0 & \text{if } x - ct < -2 \text{ and } x + ct < -2, \\ \frac{1}{2c} (2 + x + ct) & \text{if } x - ct < -2 \text{ and } -2 \leq x + ct \leq 2, \\ \frac{2}{c} & \text{if } x - ct < -2 \text{ and } 2 < x + ct, \\ t & \text{if } -2 \leq x - ct \text{ and } x + ct \leq 2, \\ \frac{1}{2c} (2 - x + ct) & \text{if } -2 \leq x - ct \leq 2 \text{ and } 2 < x + ct, \\ 0 & \text{if } 2 < x - ct \text{ and } 2 < x + ct. \end{cases}
\]

Solution at \( t = 0 \) is identically 0.

---

Figure 0.5: Plots of solution at times 1, 2, 3, 4, 5, and 6.
A spherical wave is a solution of the three-dimensional wave equation of the form $u(r,t)$ where $r$ is the distance from the origin. The wave equation takes the form

$$u_{tt} = c^2 \left( u_{rr} + \frac{2}{r} u_r \right).$$

which is called the spherical wave equation.

a) Let $v = ru$ to get the equation for $v$: $v_{tt} = c^2 v_{rr}$.

b) Solve for $v$ using the general solution of the wave equation and thereby solve the spherical wave equation (1).

c) Use the D'Alembert solution of the wave equation to solve Eq(1) with initial conditions $u(r,0) = f(r)$ and $u_t(r,0) = g(r)$. Assume that both $f(r)$ and $g(r)$ are even functions of $r$.

a) Let $v = ru$. Then $u_{tt} = (1/r)v_{tt}$, $u_r = (1/r)v_r - (v/r^2)$, and $u_{rr} = (1/r^2)(rv_{rr} - v_r) - (1/r^4)(r^2v_r - 2rv)$. Hence, $u_{rr} + (2/r)u_r = v_{rr}$, and $v(r,t)$ therefore satisfies the equation $v_{tt} = c^2 v_{rr}$.

b) Consider $v_{tt} = c^2 v_{rr}$ for all $r$. The solution has the form $v(r,t) = F(r - ct) + G(r + ct)$ for arbitrary functions $F$ and $G$. Hence,

$$u(r,t) = \frac{1}{r}F(r - ct) + \frac{1}{r}G(r + ct).$$

c) Since $u(r,0) = f(r)$, $v(r,0) = rf(r)$. Since $u_t(r,0) = g(r)$, $v_t(r,0) = rg(r)$. The problem $v_{tt} = c^2 v_{rr}$ with these initial conditions has the solution

$$v(r,t) = \frac{1}{2} \{(r - ct)f(r - ct) + (r + ct)f(r + ct)\} + \frac{1}{2c} \int_{r-ct}^{r+ct} sg(s)ds.$$

Since $u = v/r$, the solution $u(r,t)$ is

$$u(r,t) = \frac{1}{2r} \{(r - ct)f(r - ct) + (r + ct)f(r + ct)\} + \frac{1}{2cr} \int_{r-ct}^{r+ct} sg(s)ds.$$

Since $u(r,t) = v(r,t)/r$, there is a problem as $r \to 0$, unless $v(r,t) = 0$. But the fact that $f(r)$ and $g(r)$ are even functions implies that $rf(r)$ and $rg(r)$ are odd functions. Therefore,

$$v(0,t) = \frac{1}{2} \{-ctf(-ct) + ctf(ct)\} + \frac{1}{2c} \int_{-ct}^{+ct} sg(s)ds = 0,$$

as required.
10: Solve the problem $4u_{tt} + 3u_{xt} - u_{xx} = 0$ with $u(x, 0) = x^2$ and $u_t(x, 0) = e^x$. (Hint: One way to do this is to factor the PDE. Another is to use a transformation based on the characteristic coordinates.)

Write the PDE in factored form as

$$
\left( \frac{4}{\tau} \frac{\partial}{\partial \xi} - \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial x} \right) u = 0.
$$

Let $v = u_t + u_x$. We want $4v_t - v_x = 0$, or $v_t - 1/4v_x = 0$. This implies that $v(x,t) = F(x + t/4)$ for any function $F$. We then want

$$
u_t + u_x = F(x + t/4).
$$

We look for a particular solution to this in the form $u(x,t) = H(x + 1/4t)$. Since $u_t = H'/4$ and $u_x = H'$, we get a particular solution by choosing $H$ such that $5/4H' = F$. Since $F$ is arbitrary, so is $H$. The general solution of the nonhomogeneous equation (2) for $u$ is this particular solution plus the general solution of the corresponding homogeneous equation, so

$$
u(x,t) = H(x + t/4) + G(x - t)
$$

for arbitrary functions $H$ and $G$. To satisfy the initial conditions we want

$$
x^2 = u(x,0) = H(x) + G(x),
$$

and

$$
e^x = u_t(x,0) = 1/4H'(x) - G'(x).
$$

Differentiating the first of these two equations gives us two equations for $H'$ and $G'$, namely

$$
\begin{align*}
H' + G' &= 2x \\
1/4H' - G' &= e^x
\end{align*}
$$

Hence $5/4H'(x) = 2x + e^x$ and $H(x) = 4/5(x^2 + e^x) + C$ for constant $C$. Then $G(x) = x^2 - H(x) = 1/5x^2 - 4/5e^x - C$. So the solution $u(x,t)$ is given by

$$
u(x,t) = H(x + 1/4t) + G(x - t) = 4/5 \left( (x + t/4)^2 + e^{x+t/4} \right) + 1/5 \left( (x - t)^2 - 4e^{x-t} \right).
$$

Another way to solve the problem is to make a change of variables: Let $\xi = x + bt$, $\tau = x + dt$, and $U(\xi, \tau) = u(x,t)$. We then find that the PDE can be written

$$
(4b^2 + 3b - 1)U_{\xi\xi} + (4d^2 + 3d - 1)U_{\tau\tau} + (8bd + 3(b + d) - 2)U_{\xi\tau} = 0.
$$

Choosing $b = (-3 + \sqrt{25})/8 = 1/4$ and $d = (-3 - \sqrt{25})/8 = -1$, the coefficients of the $U_{\xi\xi}$ and $U_{\tau\tau}$ terms are 0, and the coefficient of the $U_{\xi\tau}$ term is nonzero. So the equation becomes $U_{\xi\tau} = 0$. This has solution $U(\xi, \tau) = H(\xi) + G(\tau)$ for arbitrary differentiable functions $H$ and $G$. Hence $u(x,t) = H(x + t/4) + G(x - t)$. Then $H$ and $G$ are determined from the initial conditions as above.