Name: 

MATH 1180
MATHEMATICS FOR LIFE SCIENTISTS
Midterm 2, March 28, 2003, ANSWERS

Do the following problems. Each problem is worth the number of points indicated. Write readable answers on the test but feel free to use and hand in extra paper. You may use the one 8 1/2 inch by 11 inch sheet of notes you prepared and your calculator. BE SURE TO ANSWER ALL PARTS OF EACH QUESTION.

1) (25 points) Suppose a population obeys the rule

\[ N_{t+1} = N_t + I_t \]

with initial population size \( N_0 = 100 \). Here, immigration \( I_t \) is a random variable with the probability distribution:

\[
I_t = \begin{cases} 
10 & \text{with probability 0.40} \\
2 & \text{with probability 0.25} \\
-8 & \text{with probability 0.30} \\
-10 & \text{with probability 0.05} 
\end{cases}
\]

Determine whether the population is expected to grow or shrink over time. Guesses are not enough. Show the work that leads to your conclusion. If your answer is 'grow', about how long will it take for the population to reach 150? If your answer is 'shrink', about how long will it take for the population to decrease to 50?

**ANSWER**

\[ N_t = N_0 + I_0 + I_1 + \ldots + I_{t-1} \]

The population will grow if the expectation of \( I_t \) is positive.

\[
E(I_t) = (10)(0.4) + (2)(0.25) + (-8)(0.3) + (-10)(0.05) = 1.6
\]

so \( N_t \) grows. To increase from 100 to 150 that is by 50 is expected to take \( 50/1.6 = 31.25 \) generations.
2) (25 points) Suppose that at time 0, there is 1 toxin molecule in a cell and 1 toxin molecule outside the cell. Suppose that the probability of a molecule leaving the cell is 0.8 each second, and that the probability of a molecule outside the cell entering the cell is 0.1 each second. a) Find a random variable that counts the number of molecules in the cell at time 1. Find the probabilities of the outcomes. Find the expectation and variance of the random variable. Assume that the molecules move independently of one another. b) What is the probability that the number of molecules in the cell at time 2 is 2?

**ANSWER** Let \( N_1 \) be the number of molecules in the cell at time \( t = 1 \). The possible values of \( N_1 \) are 0, 1, 2. Call the molecule that is initially inside the cell molecule ‘a’, and the molecule that is initially outside the cell molecule ‘b’. Let \( I_a \) be the event that molecule ‘a’ is inside at \( t = 1 \), and let \( I_b \) be the event that molecule ‘b’ is inside at \( t = 1 \).

\[
\Pr(N_1 = 2) = \Pr(I_a \cap I_b) = \Pr(I_a) \cdot \Pr(I_b) = (0.2)(0.1) = 0.02
\]

\[
\Pr(N_1 = 0) = \Pr(I_a^c \cap I_b^c) = \Pr(I_a^c) \cdot \Pr(I_b^c) = (0.8)(0.9) = 0.72
\]

\[
\Pr(N_1 = 1) = 1 - \Pr(N_1 = 0) - \Pr(N_1 = 2) = 0.26
\]

\[
E(N_1) = (0)(0.72) + (1)(0.26) + (2)(0.02) = 0.30
\]

\[
E(N_1^2) = (0)(0.72) + (1^2)(0.26) + (2^2)(0.02) = 0.34
\]

\[
\Var(N_1) = E(N_1^2) - [E(N_1)]^2 = 0.34 - (0.3)^2 = 0.25
\]

To calculate the probability that two molecules are inside the cell at time \( t = 2 \), let \( N_2 \) be the number of cells inside at \( t = 2 \), let \( I_{a,1} \) be the event that molecule ‘a’ is inside the cell at time \( t = 1 \), let \( I_{a,2} \) be the event that molecule ‘a’ is inside the cell at time \( t = 2 \), let \( I_{b,1} \) be the event that molecule ‘b’ is inside the cell at time \( t = 1 \), and let \( I_{b,2} \) be the event that molecule ‘b’ is inside the cell at time \( t = 2 \).

\[
\Pr(N_2 = 2) = \Pr(I_{a,2} \cap I_{b,2}) = \Pr(I_{a,2}) \cdot \Pr(I_{b,2})
\]

\[
= \left[ \Pr(I_{a,2} | I_{a,1}) \Pr(I_{a,1}) + \Pr(I_{a,2} | I_{a,1}^c) \Pr(I_{a,1}^c) \right] \cdot \left[ \Pr(I_{b,2} | I_{b,1}) \Pr(I_{b,1}) + \Pr(I_{b,2} | I_{b,1}^c) \Pr(I_{b,1}^c) \right]
\]

\[
= \left[ (0.2)(0.2) + (0.1)(0.8) \right] \left[ (0.2)(0.1) + (0.1)(0.9) \right] = [0.12][0.11] = 0.0132
\]
3) (25 points) Suppose that a particular gene has a 2% chance of mutating each time a cell divides and that there is a 1% chance of correcting a mutant each time the cell divides. So at any time \( t \), the gene is either a mutant or not. This can be described by a two-state Markov chain where the event \( M_t \) occurs if the gene is a mutant at time \( t \), and the event \( N_t \) occurs if the gene is not a mutant at time \( t \). 

**a)** Use conditional probabilities to describe this two-state Markov chain, that is the probabilities of the gene being mutant or not at time \( t+1 \) depending on whether it is mutant or not at time \( t \). 

**b)** Find an updating function for the probability that the gene is mutant after \( t \) cell divisions. 

**c)** Find the long-term (equilibrium) probability that the gene is mutant.

**ANSWER**

**a)** \( \Pr(M_{t+1}|M_t) = 0.99 \), \( \Pr(N_{t+1}|M_t) = 0.01 \), \( \Pr(M_{t+1}|N_t) = 0.02 \), and \( \Pr(N_{t+1}|N_t) = 0.98 \).

**b)** Let \( p_t = \Pr(M_t) \) be the probability the gene is mutant at time \( t \). Using the Law of Total Probability, we have

\[
p_{t+1} = \Pr(M_{t+1}) = \Pr(M_{t+1}|M_t)\Pr(M_t) + \Pr(M_{t+1}|N_t)\Pr(N_t) \\
= 0.99p_t + 0.02(1 - p_t) \\
p_{t+1} = 0.97p_t + 0.02
\]

**c)** The equilibrium probability \( p^* \) satisfies \( p^* = 0.97p^* + 0.02 \), so \( p^* = 2/3 \).
4) (25 points) Consider a continuous random variable $X$ which takes values between 0 and 1 and has the pdf $f(x) = 3x^2$. 

a) Graph $f$ and show that it is a valid pdf. 

b) Compute the cdf for $X$. 

c) Compute the expectation of $X$ and mark it on your graph. 

d) Compute the mode of $X$ and mark it on your graph. 

e) Compute the median of $X$ and mark it on your graph. 

f) Compute the variance of $X$. 

**ANSWER**

![Graph of the pdf $f(x) = 3x^2$]

a) The graph is shown.

$$\int_0^1 3x^2 \, dx = x^3 \bigg|_0^1 = 1$$

Since $f(x) \geq 0$ and has integral 1, it is a valid pdf.

b) $F(x) = \int_0^x f(s) \, ds = s^3 \bigg|_0^1 = x^3$.

c) $E(X) = \int_0^1 xf(x) \, dx = 3 \int_0^1 x^3 \, dx = 3/4$.

d) The mode is where the highest value of the pdf occurs. It is $x = 1$.

e) The median occurs where $F(x) = 1/2$. Solving $x^3 = 1/2$ gives $x = 2^{-1/3} \approx 0.79$.

f) $E(X^2) = \int_0^1 x^2f(x) \, dx = 3 \int_0^1 x^4 \, dx = 3/5$. So $Var(X) = E(X^2) - [E(X)]^2 = 3/5 - 9/16 = 3/80$. 