4.3 Solving Pure-Time Differential Equations

MATHEMATICAL TECHNIQUES

 indeb From the graphs, sketch the antiderivative of the function that passes through the given point.

• EXERCISE 4.3.1
  The antiderivative should pass through the point (0, 500)

  ![Graph](image)

• EXERCISE 4.3.2
  The antiderivative should pass through the point (50, 5000)

  ![Graph](image)

• EXERCISE 4.3.3
  The antiderivative should pass through the point (0, 2500).

  ![Graph](image)

• EXERCISE 4.3.4
  The antiderivative should pass through the point (100, 3000).

  ![Graph](image)

• EXERCISE 4.3.5
  The antiderivative should pass through the point (100, 1000).

  ![Graph](image)
Exercise 4.3.6
The antiderivative should pass through the point (0, 1000).

- Exercise 4.3.7
  \[ 7t^2. \]
- Exercise 4.3.8
  \[ 10t^6 + 6t^5. \]
- Exercise 4.3.9
  \[ 72t + 5. \]
- Exercise 4.3.10
  \[ y^4 + 5y^3. \]
- Exercise 4.3.11
  \[ \frac{5}{x^3}. \]
- Exercise 4.3.12
  \[ 3z^4. \]
- Exercise 4.3.13
  \[ \frac{2}{\sqrt{t}} + 3. \]
- Exercise 4.3.14
  \[ 5z^{-1.2} - 1.2. \]

Use antiderivatives to solve the following differential equations. Sketch a graph of the rate of change and the solution on the given domain.

- Exercise 4.3.15
  \[ \frac{dV}{dt} = 2t^2 + 5 \text{ with } V(1) = 19.0. \] Sketch the rate of change and solution for \( 0 \leq t \leq 5. \)
- Exercise 4.3.16
  \[ \frac{dV}{dt} = 2t^2 + 5 \text{ with } V(0) = 19.0. \] Sketch the rate of change and solution for \( 0 \leq t \leq 5. \)
- Exercise 4.3.17
  \[ \frac{df}{dt} = 5t^3 + 5t \text{ with } f(0) = -12.0. \] Sketch the rate of change and solution for \( 0 \leq t \leq 2. \)
- Exercise 4.3.18
  \[ \frac{dg}{dt} = -3t + t^2 \text{ with } g(0) = 10.0. \] Sketch the rate of change and solution for \( 0 \leq t \leq 5. \)
- Exercise 4.3.19
  \[ \frac{dM}{dt} = t^2 + \frac{1}{t^2} \text{ with } M(3) = 10.0. \] Sketch the rate of change and solution for \( 0 \leq t \leq 3. \)
- Exercise 4.3.20
  \[ \frac{dp}{dt} = 5t^3 + \frac{2}{t^2} \text{ with } p(1) = 12.0. \] Sketch the rate of change and solution for \( 1 \leq t \leq 3. \)
4.3. SOLVING PURE-TIME DIFFERENTIAL EQUATIONS

There are no simple integral versions of product and quotient rules for derivatives. Use the given functions to show that proposed rule does not work.

- **EXERCISE 4.3.21**
  Use the functions $f(x) = x^2$ and $g(x) = x^3$ to show that the product of integrals is not equal to the integral of the product.

- **EXERCISE 4.3.22**
  Use the functions $f(x) = x^2$ and $g(x) = x^3$ to show that $\int f(x)g(x)dx \neq g(x)\int f(x)dx + f(x)\int g(x)dx$.

**APPLICATIONS**

Suppose a cell is taking water into two vacuoles. Let $V_1$ denote the volume of the first vacuole and $V_2$ the volume of the second. In each of the following cases,

- **a.** Solve the given differential equations for $V_1(t)$ and $V_2(t)$.
- **b.** Write a differential equation for $V$, including the initial condition.
- **c.** Show that the solution of the differential equation for $V$ is the sum of the solutions for $V_1$ and $V_2$.

- **EXERCISE 4.3.23**

\[
\begin{align*}
d\frac{V_1}{dt} & = 2.0t + 5.0 \\
d\frac{V_2}{dt} & = 5.0t + 2.0
\end{align*}
\]

with initial conditions $V_1(0) = V_2(0) = 10\mu m^3$.

- **EXERCISE 4.3.24**

\[
\begin{align*}
d\frac{V_1}{dt} & = 3.6t^2 + 5.0t \\
d\frac{V_2}{dt} & = 5.2t^3 + 2.0
\end{align*}
\]

with initial conditions $V_1(0) = 5.0\mu m^3$ and $V_2(0) = 10.0\mu m^3$.

Suppose organisms grow in mass according to the differential equation

\[
d\frac{M}{dt} = \alpha t^n
\]

where $M$ is measured in grams and $t$ is measured in days. For each of the following values for $n$ and $\alpha$, find

- **a.** The units of $\alpha$.
- **b.** Suppose that $M(0) = 5.0\text{gm}$. Find the solution.
- **c.** Sketch a graph of the rate of change and the solution.
- **d.** Describe your results in words.

- **EXERCISE 4.3.25**
  $n = 1, \alpha = 2.0$.

- **EXERCISE 4.3.26**
  $n = -1/2, \alpha = 2.0$.

In a new program, NASA sends unmanned expeditions to astronomical bodies. One of the more fascinating experiments involves having the robot release an object from a height $h = 100\text{m}$ with velocity $v = 5.0\text{m/sec}$ (upward) to find its trajectory in the local gravitational field of strength $a$. For the following values of $a$,

- **a.** Find the velocity and position of the object as functions of time.
b. How high will the object get?

c. How long will it take to pass the robot on the way down? How fast will it be moving?

d. How long will it take to hit the ground? How fast will it be moving? How fast is this in miles per hour?

e. Graph the velocity and position as functions of time.

• EXERCISE 4.3.27
  A practice test on earth where \( a = -9.8 \text{m/s}^2 \).

• EXERCISE 4.3.28
  On the moon where \( a = -1.62 \text{m/s}^2 \).

• EXERCISE 4.3.29
  On Jupiter where \( a = -22.88 \text{m/s}^2 \).

• EXERCISE 4.3.30
  On Mars' moon Deimos where \( a = 2.15 \times 10^{-3} \text{m/sec}^2 \).

The velocities of four objects are measured at discrete times.

<table>
<thead>
<tr>
<th>time</th>
<th>velocity of object 1</th>
<th>velocity of object 2</th>
<th>velocity of object 3</th>
<th>velocity of object 4</th>
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<tbody>
<tr>
<td>0</td>
<td>1.0</td>
<td>9.0</td>
<td>25.0</td>
<td>1.0</td>
</tr>
<tr>
<td>1</td>
<td>3.0</td>
<td>7.0</td>
<td>16.0</td>
<td>3.0</td>
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<td>9.0</td>
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<td>4.0</td>
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</tr>
<tr>
<td>4</td>
<td>9.0</td>
<td>1.0</td>
<td>1.0</td>
<td>15.0</td>
</tr>
</tbody>
</table>

Use Euler’s method to estimate the position at \( t = 4 \) starting from the given initial condition, and find a formula for the velocities and use to find the exact position at \( t = 4 \). Graph your results.

• EXERCISE 4.3.31
  Object 1, with initial condition \( p(0) = 10.0 \). To find the formula for the velocities, note that they increase linearly in time.

• EXERCISE 4.3.32
  Object 2, with initial condition \( p(0) = 10.0 \). To find the formula for the velocities, note that they decrease linearly in time.

• EXERCISE 4.3.33
  Object 3, with initial condition \( p(0) = 10.0 \). To find the formula for the velocities, compare them with the perfect square numbers.

• EXERCISE 4.3.34
  Object 4, with initial condition \( p(0) = 10.0 \). The velocities follow a quadratic equation of the form \( v(t) = \frac{t^2}{2} + at + b \) for some value of \( a \).

Consider again the velocities of four objects used in the previous set of problems. There is a more accurate variant of Euler’s method that approximates the rate of change during a time interval as the average of the rate of change at the beginning and at the end of the interval. For example, if \( v(0) = 1.0 \) and \( v(1) = 3.0 \), we approximate the rate of change for \( 0 \leq t \leq 1 \) as 2.0. Use this variant of Euler’s method to estimate the position at \( t = 4 \) and compare with the exact position at \( t = 4 \). Graph your results.

• EXERCISE 4.3.35
  Object 1, with initial condition \( p(0) = 10.0 \).

• EXERCISE 4.3.36
  Object 2, with initial condition \( p(0) = 10.0 \).

• EXERCISE 4.3.37
  Object 3, with initial condition \( p(0) = 10.0 \).

• EXERCISE 4.3.38
  Object 4, with initial condition \( p(0) = 10.0 \).
Chapter 5
Answers

4.3.1.

4.3.3.

4.3.5.

4.3.7. \[ \int 7x^2 \, dx = \frac{7x^3}{3} + c. \]
4.3.9. \[ \int 7t + 5 \, dt = 36t^2 + 5t + c. \]
4.3.11. \[ \int \frac{5}{2} \, dx = -\frac{5x^2}{2} + c. \]
4.3.13. \[ \int \frac{2}{\sqrt[3]{t}} + 3 \, dt = \frac{4t^{2/3}}{3} + 3t + c. \]
4.3.15. Integrating, we find that \[ V(t) = 2t^3/3 + 5t + c. \] Substituting the initial condition, \[ V(1) = 2/3 + 5.0 + c = 19.0 \] so \[ c = 13.3. \] The solution is \[ V(t) = 2t^3/3 + 5t + 13.3. \]
4.3.17. \( f(t) = 1.25t^4 + 2.5t^2 + c. \) Substituting the initial condition, \( f(0) = c = -12.0 \) so \( c = -12. \) The solution is \( f(t) = 1.25t^4 + 2.5t^2 - 12.0. \)

4.3.19. \( M(t) = \frac{t^3}{3} - \frac{1}{4} + c. \) Substituting the initial condition, \( M(1) = \frac{27}{3} - \frac{1}{4} + c = 10.0, \) so \( c = 1.33. \)

4.3.21. Let \( F(x) = \int f(x)dx = \frac{x^3}{3} + c, \) and \( G(x) = \int g(x)dx = \frac{x^4}{4} + c. \) The product of the functions is \( f(x)g(x) = x^5, \) with integral \( \int x^5dx = \frac{x^6}{6} + c \neq F(x)G(x). \)

4.3.23.
   a. \( V_1(t) = t^2 + 5.0t + 10.0, \) \( V_2(t) = 2.5t^2 + 2.0t + 10.0. \)
   b. \( \frac{dV}{dt} = 7.0t + 7.0 \) with initial condition \( V(0) = V_1(0) + V_2(0) = 20. \)
   c. \( V(t) = 3.5t^2 + 7.0t + 20.0. \) This is indeed the sum of \( V_1(t) \) and \( V_2(t). \)

4.3.25.
   a. The units of \( \alpha \) must be \( \frac{\text{grams}}{\text{day}}. \)
   b. \( M(t) = t^2 + c. \) Substituting the initial conditions, we have \( M(0) = c = 5.0 \) so \( M(t) = t^2 + 5.0. \)
4.3.27.

a. $v(t) = -9.8t + 5.0$, $p(t) = -4.9t^2 + 5.0t + 100$.

b. The maximum height is when $v(t) = 0$, or at $t = 0.51$. The position is 101.27.

c. It passes the robot when $p(t) = 100$, or at $t = 0$ and $t = 1.02$. The velocity is -5.0 m/s.

d. It take 5.06 s to hit the ground, and will be moving at -44.55 m/s. This is 99.65 MPH.

4.3.29.

a. $v(t) = -22.88t + 5.0$, $p(t) = -11.44t^2 + 5.0t + 10$.

b. The maximum height is when $v(t) = 0$, or at $t = 0.22$. The position is 100.55.

c. It passes the robot when $p(t) = 100$, or at $t = 0$ and $t = 0.44$. The velocity is -5.0 m/s.

d. It take 3.18 s to hit the ground, and will be moving at -67.83 m/s. This is 151.7 MPH.

4.3.31. With Euler’s method, $\dot{p}(1) = 11.0$, $\dot{p}(2) = 14.0$, $\dot{p}(3) = 19.0$, $\dot{p}(4) = 26.0$. The velocities fall on the line $v(t) = 1.0 + 2.0t$. Therefore, position satisfies the differential equation $\frac{dp}{dt} = 1.0 + 2.0t$. This has solution $p(t) = 1.0t + t^2 + 10.0$ when $p(0) = 10$. At $t = 4$, the exact solution is $p(t) = 30.0$.

4.3.33. With Euler’s method, $\dot{p}(1) = 35.0$, $\dot{p}(2) = 51.0$, $\dot{p}(3) = 60.0$, $\dot{p}(4) = 64.0$. The velocities fall on the quadratic $v(t) = (t - 5.0)^2 = t^2 - 10.0t + 25.0$. The position satisfies the differential equation $\frac{dp}{dt} = t^2 - 10t + 25$. This has solution $p(t) = \frac{t^3}{3} - 5.0t^2 + 25.0t + 10.0$ when $p(0) = 10$. At $t = 4$, the exact solution is $p(t) = 51.333$. 

d. The mass increases more and more quickly.
4.3.35. We estimate the velocity to be 2.0 during the first minute, 4.0 during the second, 6.0 during the third, and 8.0 during the fourth. Then \( \dot{p}(1) = 12.0, \dot{p}(2) = 16.0, \dot{p}(3) = 22.0, \dot{p}(4) = 30.0 \). At \( t = 4 \), this exactly matches the exact solution of \( p(t) = 30.0 \).

4.3.37. We estimate the velocity to be 20.5 during the first minute, 12.5 during the second, 6.5 during the third, and 2.5 during the fourth. Then \( \dot{p}(1) = 30.5, \dot{p}(2) = 43.0, \dot{p}(3) = 49.5, \dot{p}(4) = 52.0 \). At \( t = 4 \), the exact solution is \( p(t) = 51.333 \), so this is much closer.